2017 SCSU MATH CONTEST
11^{th} and 12^{th} Grade Test—50^{th} Annual Edition

DIRECTIONS: Select the BEST response from among those given. Scientific and graphing calculators are allowed. Symbolic calculators are not allowed.

1. The first SCSU Math Contest was held in 1968. In honor of this, and in honor of the fact that some mathematicians are a bit odd, find the sum of the ODD integers from 1968 through 2017 (including 2017).

   (a) 47818 (b) 49800 (c) 49825 (d) 51792 (e) 51817

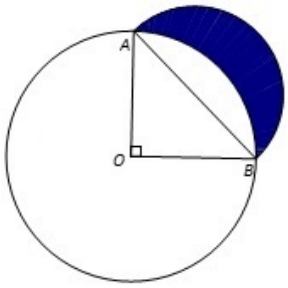
2. The sum of two prime numbers is 199. What is the product of these two primes?

   (a) 143 (b) 394 (c) 637 (d) 2101 (e) 9797

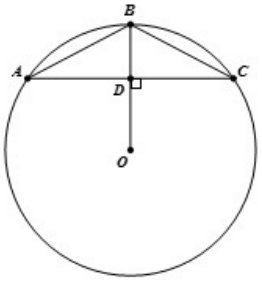
3. A circle has center O. Segments OA and OB are radii, and $\angle AOB$ is a right angle, as shown in the diagram below left. A semicircle is constructed using segment AB as its diameter.

   The shaded moon-shaped portion of the semicircle outside the original circle is called a *lune*. What is the ratio of the area of the lune to the area of triangle $\triangle AOB$?

   (a) $\sqrt{2} : \pi$ (b) $1 : 1$ (c) $\frac{\pi}{3} : 1$ (d) $\frac{\pi}{\sqrt{3}} : 1$ (e) $\pi : \sqrt{2}$



Problem #3



Problem #4

4. In the figure above right, the radius of the circle is $OC = 1$, and OB is perpendicular to AC. Let x denote the length of segment OD. Express the area of $\triangle ABC$ as a function of x.

   (a) $(1-x)\sqrt{1-x^2}$ (b) $\frac{1}{2}x(1-x)$ (c) $\frac{1}{2}\sqrt{1-x^2}$ (d) $\frac{1}{2}$ (e) $\frac{1}{2x\sqrt{x^2-1}}$

5. I'm thinking of my two favorite numbers. Their *sum* is a, and their *product* is b. What is the sum of the *reciprocals* of my two favorite numbers?

   (a) $\frac{a+b}{ab}$ (b) $\frac{a^2+2ab+b^2}{ab}$ (c) $\frac{a+b}{a^2+b^2}$ (d) $\frac{a}{b}$ (e) $\frac{ab}{a+b}$

6. Suppose that the function f is given by $f(x) = \frac{x^2-7x+10}{x-4}$. Find the equations of any horizontal, vertical, and/or slant asymptotes of the graph of f.

   (a) $x = 4$ only (b) $x = 4$ and $y = 0$ (c) $x = 4$ and $y = x-3$ (d) $x = 4$ and $y = x-7$ (e) $x = 4$, $y = 2$, $y = 5$

7. How many triangles have area 20 and vertices $(-17, 0)$, $(17, 0)$, and $(17\cos\theta, 17\sin\theta)$ for some θ?

   (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

8. Three tennis balls come in a standard cylindrical can. The radius of each ball, and of the can itself, is 3.5 centimeters. To the nearest whole percentage point, how much of the can is NOT occupied?

   (a) 8% (b) 14% (c) 25% (d) 33% (e) 50%

9. Write the quotient in standard $a + bi$ form: $\frac{12+8i}{6-2i}$.

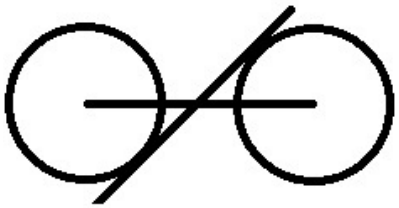
   (a) 5 (b) $2 - 4i$ (c) $-6 + \frac{4}{3}i$ (d) $\frac{11}{4} + \frac{3}{4}i$ (e) $\frac{7}{5} + \frac{9}{5}i$

10. A drawer contains four brown socks, four blue socks, and four white socks. How many socks must be pulled from the drawer at random to be sure of having **two** pairs?

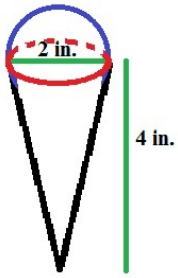
(a) 5 (b) 6 (c) 7 (d) 8 (e) 9

11. Two circles have radius 3 inches, and their centers are 10 inches apart. A line is tangent to the circles, as shown in the diagram below left. What is the distance (in inches) between the points at which the tangent line touches the circles?

(a) 6 (b) 8 (c) 9 (d) 10 (e) 12



Problem #11



Problem #12

12. An ice cream cone is four inches tall, and its top has a diameter of two inches, as in the diagram above right. The entire cone is filled with ice cream, and it is topped with a hemisphere of ice cream with a diameter of two inches. What is the total volume of ice cream, in cubic inches?

(a) $\frac{4}{3}\pi$ (b) $\frac{3}{2}\pi$ (c) $\frac{5}{3}\pi$ (d) 2π (e) $\frac{14}{3}\pi$

13. The graph of the function $f(x) = ax^2 + bx + c$ passes through the points $(1, 8)$, $(2, 0)$, and $(3, 16)$. What is $f(x)$?

(a) $f(x) = 2x^2 + 3x - 14$ (b) $f(x) = 3x^2 - 11x + 10$ (c) $f(x) = x^2 - 4x + 4$

(d) $f(x) = 12x^2 - 44x + 40$ (e) $f(x) = 6x^2 + 22x - 10$

14. Let $f(x) = x^2 - 7x + 10$. For what values of t do we have $f(t - 4) = 0$?

(a) 1 and 6 (b) 2 and 5 (c) 3 and 7 (d) 5 and 8 (e) 6 and 9

15. How many three-digit *even* positive integers have three *distinct* digits? (Zero cannot be the first digit.)

(a) 324 (b) 328 (c) 332 (d) 336 (e) 340

16. The product of the digits of a two-digit number plus the sum of the digits is equal to the two-digit number itself. What is the units digit of the number?

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

17. A cube has 3-inch edges. A fly lands at one of the vertices and then walks along the edges. What is the greatest distance (in inches) the fly could travel without walking to any vertex a second time or retracing any part of its path?

(a) 12 (b) 18 (c) 24 (d) 30 (e) 36

18. Suppose that you randomly permute the letters in `FIFTIETH`. What is the probability that the letters `HIT` will appear consecutively (and in that order)?

(a) 1/84 (b) 1/14 (c) 1/8 (d) 3/8 (e) 1/2

19. An equilateral triangle of side length 3 centimeters is inscribed in a circle. Find the total area of the region that lies inside the circle but outside the triangle.

(a) $3\pi - 9\sqrt{3}/4$ (b) $2\pi - \sqrt{2}/3$ (c) $3\pi - 7\sqrt{3}/4$ (d) $4\pi - 7\sqrt{3}/2$ (e) $4\pi - 5\sqrt{2}/3$

20. What is the greatest common divisor of 10^{10} and $26!$?

(a) $2^{10} \cdot 5^5$ (b) $2^4 \cdot 10^6$ (c) $2^{13} \cdot 5^6$ (d) 10^{10} (e) $2^{17} \cdot 10^6$

21. A circle has center $(-5, 6)$ and radius 3. Which of the following is a correct equation for the circle?

(a) $x^2 + 10x + 25 + y^2 - 12y + 36 = 9$ (b) $x^2 + 10x + 25 + y^2 - 12y + 36 = 3$ (c) $x^2 + 25 + y^2 - 12y + y^2 = 9$

(d) $x^2 - 10x + 25 + y^2 + 12y + 36 = 9$ (e) $x^2 - 10x + 25 + 12y + 36 = 3$

22. Suppose that $\log_x \left(2x^2 + x - 1\right) > \log_x 2 - 1$. Which of the following gives the possible values for x?

(a) $x > \frac{1}{2}$ and $x \neq 1$ (b) $\frac{1}{2} < x < 1$ (c) $0 < x < 1$ (d) $1 < x < 2$ (e) $x > 1$

23. On the island of New Smullyania (population somewhere between 1919 and 2017), every inhabitant is either a *knight* (and always tells the *truth*) or a *knave* (and always *lies*). One day, you meet three inhabitants of New Smullyania and mention that you took part in the 50th Annual SCSU Math Contest.

Adelaide proudly makes a single (though compound) statement: "I took part in the *first* SCSU Math Contest, and I placed in the top 10% for my grade."

Bernadette scowls and grumbles, "Adelaide is a knave."

Cordelia smiles and makes her own single statement: "Adelaide did take part in the first SCSU Math Contest, but she didn't place in the top 10% for her grade."

Bernadette scowls even more fiercely and mutters under her breath, "We're *all* knaves."

Which of the following can we conclude from this exchange?

(a) Adelaide is a knight; Bernadette and Cordelia are knaves.

(b) Adelaide is a knave; Bernadette and Cordelia are knights.

(c) Adelaide and Bernadette are knaves; Cordelia is a knight.

(d) All three are knaves.

(e) Bernadette is a knave; Cordelia is a knight; it is impossible to determine what type Adelaide is.

24. A rocket is launched straight upward from the ground. A few moments after the launch, an observer located at ground level 1000 feet from the base of the launchpad measures that the angle of elevation to the bottom of the rocket is $20°$ and the angle of elevation to the top of the rocket is $27°$. To the nearest *TENTH* of a foot, how long is the rocket?

(a) 112.0 (b) 145.6 (c) 230.4 (d) 364.0 (e) 509.5

25. An increasing sequence 3, 15, 24, ... is formed by the positive multiples of 3 that are one less than a perfect square. Find the 50th term of the sequence.

(a) 2499 (b) 3720 (c) 5475 (d) 5775 (e) 6240

26. What is the value of the variable p after the following psuedocode is executed?

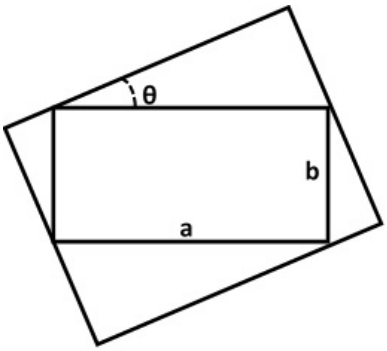
```
p := 0
for k = 2 to 3
    for m = 1 to k
        if (k+m) = 4
            p := p - 1
        else
            p := p + 2
        end if
        next m
    end inner for loop
    next k
end outer for loop
```

(a) 0 (b) 1 (c) 2 (d) 3 (e) 4

27. Let f be a function for which $f(x/3) = x^2 + x + 1$ for all x. Find the *sum* of all values of z for which $f(3z) = 7$.

(a) $-\frac{1}{3}$ (b) $-\frac{1}{9}$ (c) 0 (d) $\frac{5}{9}$ (e) 1

28. The figure below shows a rectangle with sides a and b that is circumscribed by another rectangle of area A. Define the acute angle θ as shown. Which of the following is a valid expression for A in terms of a, b, and θ?



(a) $(a^2+b^2)(a\sin\theta + b\sin\theta)$

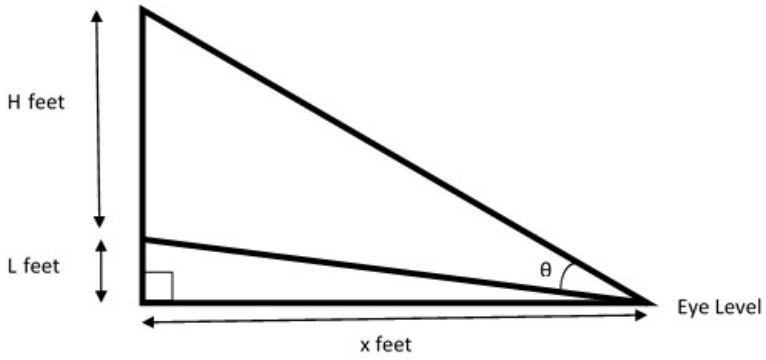
(b) $ab + a\sin(\theta) + b\cos(\theta)$

(c) $ab + a^2\sin(\theta) + b^2\cos(\theta)$

(d) $ab + \dfrac{a^2+b^2}{2}\sin(2\theta)$

(e) $ab + (a^2+b^2)\sin(2\theta)$

29. A movie theater has a screen of height H feet, and the bottom of the screen is located L feet above your eye level. Which of the following correctly expresses the viewing angle θ in terms of the distance x that you sit from the screen?



(a) $\sin^{-1}\left(\dfrac{H}{\sqrt{x^2+(H+L)^2}}\right)$

(b) $\tan^{-1}\left(\dfrac{H+L}{x}\right) - \tan^{-1}\left(\dfrac{L}{x}\right)$

(c) $\tan^{-1}\left(\dfrac{H}{\sqrt{x^2+L^2}}\right)$

(d) $\tan^{-1}\left(\dfrac{H}{x}\right)$

(e) $\tan^{-1}\left(\dfrac{H+L}{x}\right) + \tan^{-1}\left(\dfrac{L}{x}\right)$

30. An equilateral triangle is inscribed inside a unit circle. Two points are randomly chosen on the unit circle and connected to form a chord. What is the probability that the length of the chord exceeds the side length of the triangle?

(a) 1/8 (b) 1/6 (c) 1/4 (d) 1/3 (e) 1/2

31. What is the units digit of the decimal representation of $1^1 + 3^3 + 5^5 + \cdots + 2017^{2017}$?

(a) 1 (b) 3 (c) 5 (d) 7 (e) 9

32. A fair six-sided die, marked with the integers from 1 to 6, is rolled three times. The three numbers obtained are added. To the nearest *THOUSANDTH*, what is the probability that the sum is a prime number?

(a) 0.125 (b) 0.282 (c) 0.319 (d) 0.338 (e) 0.375

33. A triangle has sides of length 5, 6, and 7. A rectangle is positioned inside the triangle, with an edge on one side of the triangle and one vertex on each of the other two sides of the triangle.

What is the largest possible area of the rectangle?

(a) $4\sqrt{3}$ (b) $5\sqrt{2}$ (c) $3\sqrt{6}$ (d) $4\sqrt{5}$ (e) $3\sqrt{10}$

34. Let z be a complex number, and denote its conjugate as $\overline{z}$. Suppose that $(12+5i)z + (12-5i)\overline{z} = 26$. What is the smallest possible value of the modulus $|z|$?

(a) $\dfrac{1}{2}$ (b) 1 (c) $\sqrt{2}$ (d) $\sqrt{3}$ (e) 2

35. Find the non-real zeroes of $f(x) = 20x^4 - 148x^3 + 355x^2 - 317x + 105$. Next, add those non-real zeroes. What is the sum?

(a) 0 (b) 1.4 (c) 6 (d) 7.4 (e) 8.6