2015 SCSU MATH CONTEST
11^{th} and 12^{th} GRADE

DIRECTIONS: Select the <u>BEST</u> completion or response from among those given. Scientific and graphing calculators are allowed. Symbolic calculators are not allowed.

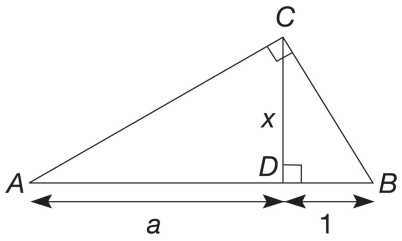
1. (Recall that "$|X|$" denotes the number of elements in the set X.) Suppose that $|A| = 85$, $|B| = 86$, $|C| = 108$, $|A \cap B| = 38$, $|A \cap C| = 27$, $|B \cap C| = 35$, and $|A \cap B \cap C| = 23$. Find $|A \cup B \cup C|$.

   (a) 156 (b) 179 (c) 202 (d) 246 (e) 279

2. The graphs of the equations $y = ax$ and $y = \dfrac{b}{x}$ have two intersection points, one of which is $(-3, -2)$. What is the other intersection point?

   (a) $(2, 3)$ (b) $(2, -3)$ (c) $(-2, 3)$ (d) $(3, 2)$ (e) $(3, -2)$

3. In the figure below, suppose we know that $a = 9$ inches.



   What is the length of x?

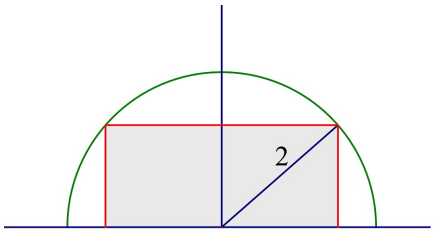
   (a) $\sqrt{3}$ (b) $\sqrt{5}$ (c) 2.5 (d) 3 (e) 4

4. Which of the following is (are) rational?

I.	2.015 2015 2015 2015 . . .
II.	0.1 01 001 0001 00001 . . .
III.	3.141592654
IV.	$(216)^{-\frac{4}{3}}$
V.	$\cos\left(\frac{25\pi}{3}\right)$

   (a) I and II (b) I, II, and III (c) III only (d) all except II (e) all except V

5. A rectangle is inscribed in a semicircle of radius 2, with center at the origin. See the figure.



   Express the area of the rectangle as a function of the x-coordinate of the corner of the rectangle in the first quadrant.

   (a) $A(x) = 2x^2$ (b) $A(x) = 4x^2$ (c) $A(x) = 2\sqrt{2}x^2$ (d) $A(x) = x\sqrt{4 - x^2}$ (e) $A(x) = 2x\sqrt{4 - x^2}$

6. Suppose that $g(x) = 1 - x^2$ and $f(g(x)) = \dfrac{1-x^2}{x^2}$. Which of the following is equal to $f\left(\dfrac{1}{3}\right)$?

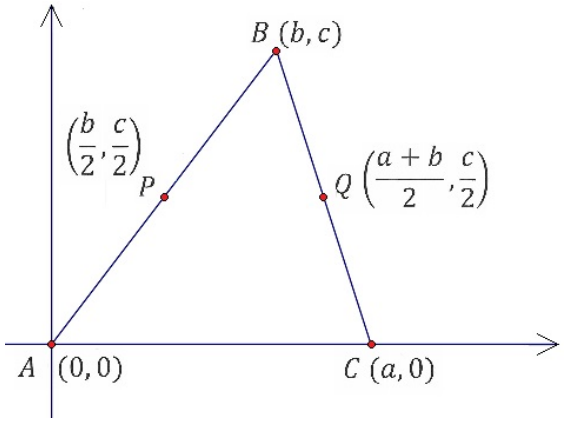
(a) $\dfrac{1}{2}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{\sqrt{6}}{3}$ (d) 1 (e) 2

7. The height of a stone thrown straight upward from a cliff is given by $h(t) = -16t^2 + 68t + 50$, where height is given in feet and time in seconds.

Find the average velocity of the stone (in *feet per second*) for t between 2 and 2.5 seconds.

(a) -8 (b) -4 (c) -1 (d) 2 (e) 4

8. Consider the figure below.



The segments $\overline{CP}$ and $\overline{AQ}$ (which are not drawn into the figure) intersect in a point M. Which one of the following is the correct value of the ratio $\dfrac{AM}{MQ}$?

(a) $\sqrt{2}$ (b) $\sqrt{3}$ (c) 2 (d) 2.5 (e) $2\sqrt{2}$

9. Simplify: $a + \dfrac{1}{b + \frac{1}{a+b}}$.

(a) $\dfrac{2a+b}{b+1}$

(b) $\dfrac{a^2+ab^2-ab}{a^2+ab+1}$

(c) $\dfrac{a^2b+ab^2+2a+b}{ab+b^2+1}$

(d) $\dfrac{a^3+3a^2b+3ab^2+1}{a^2+ab+b^2+1}$

(e) $\dfrac{a^2b+ab+b}{a^2b+ab^2+a+b+1}$

10. Suppose that N is a positive integer such that

$$\log_2(\log_3(\log_5(\log_7 N))) = 2015.$$

How many *different* prime factors does N have?

(a) one (b) two (c) three (d) four (e) six

11. Find the value of the sum: $\cos(0^\circ) + \cos(1^\circ) + \cos(2^\circ) + \cdots + \cos(360^\circ)$.

(a) 0 (b) 1 (c) π (d) 2π (e) 360

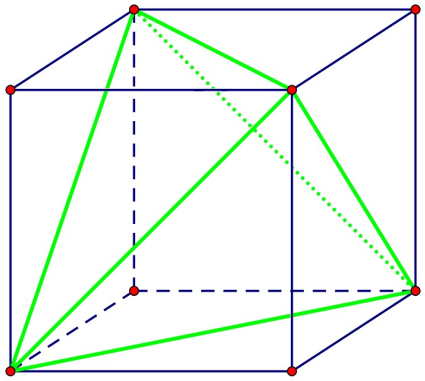
12. Solve for x: $\cos(2x) - \cos x = 0$.

(a) $\frac{\pi}{2} + k \cdot \pi$, where k is any integer (b) $\frac{2\pi}{3} \cdot k$, where k is any integer (c) $\frac{2\pi}{3} \cdot k + \pi$, where k is any integer (d) $2\pi \cdot k$, where k is any integer (e) $\pi \cdot k$ OR $\frac{2\pi}{3} \cdot k$, where k is any integer

13. How many *different* 12-letter arrangements can be made from the letters in `NEEDLESSNESS`? (Of course order matters here!)

(a) 180,180 (b) 238,100 (c) 314,640 (d) 415,800 (e) 549,480

14. Four of the eight vertices of a cube are vertices of a regular tetrahedron, as shown.



Find the ratio of the surface area of the cube to the surface area of the tetrahedron.

(a) $\frac{2\sqrt{3}}{3}$ (b) $\frac{\sqrt{6}}{2}$ (c) $\sqrt{2}$ (d) $\sqrt{3}$ (e) 2

15. The function $f(x) = x^2 - 4x + 7$ has two complex zeroes. Those two values *differ* by how much?

(a) $\frac{7}{2}$ (b) 4 (c) $2 + i\sqrt{3}$ (d) $i\sqrt{7}$ (e) $2i\sqrt{3}$

16. Suppose that the sum of the first $3n$ positive integers is 150 more than the sum of the first n positive integers. What is the sum of the first $4n$ positive integers?

(a) 300 (b) 350 (c) 400 (d) 450 (e) 600

17. Michaela found the following interesting pattern when she was solving mathematics problems:

$$\begin{aligned} 1 + 2 &= 3, \\ 4 + 5 + 6 &= 7 + 8, \\ 9 + 10 + 11 + 12 &= 13 + 14 + 15, \\ 16 + 17 + 18 + 19 + 20 &= 21 + 22 + 23 + 24, \\ &\vdots \end{aligned}$$

What is the last number in the 100th row?

(a) 10,000 (b) 10,020 (c) 10,120 (d) 10,200 (e) 10,210

18. A population grew exponentially from 1234 to 5678 in 55 years. About how many years did it take for the population to double?

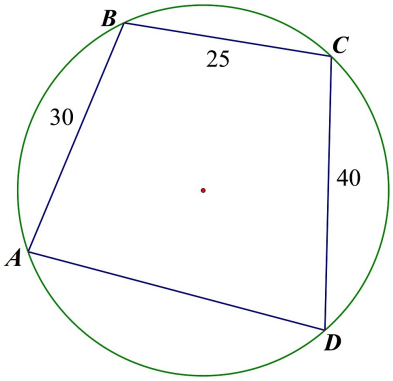
(a) 12 (b) 15 (c) 18 (d) 22 (e) 25

19. Which of the following statements is (are) true? (Assume that x and y are positive and that $y < 5$.)

I.	$(\ln x)^2 = 2\ln x$
II.	$\log_4(3x^4) = 4\log_4(3x)$
III.	$\log(x-y) = \dfrac{\log x}{\log y}$
IV.	$\log_3\left(81^5 \cdot 3^{x-20}\right) = x$
V.	$\log_5\left(\frac{125}{5-y}\right) = 3 - \log_5(5-y)$

(a) I and V only (b) I, III, and IV only (c) II and III only (d) II and V only (e) IV and V only

20. A quadrilateral is inscribed in a circle of radius 25. Three sides of the quadrilateral have lengths 30, 25, and 40, as in the figure. (*The figure is not to scale!*)



Find the length of the remaining side of the quadrilateral.

(a) 25 (b) $25\sqrt{2}$ (c) $25\sqrt{3}$ (d) 50 (e) There is not enough information to determine this.

21. A certain medical condition occurs in two varieties. Variety A affects 14% of the population; Variety B affects 20% of the population; and, thus, 66% of the population is free of the condition altogether.

Suppose that ten people are selected at random. What is the probability (rounded to the *nearest THOUSANDTH*) that exactly 1 of these ten people has Variety A, exactly 2 have Variety B, and exactly 7 do not have the condition at all?

(a) 0.110 (b) 0.169 (c) 0.259 (d) 0.365 (e) 0.516

22. A triangle is inscribed in a circle. The side lengths of the triangle are $\frac{15}{2}$, 10, and $\frac{25}{2}$.

What is the radius of the circle?

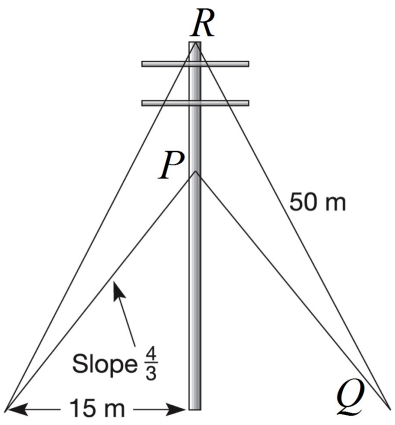
(a) $\frac{15}{4}$ (b) 5 (c) $\frac{25}{4}$ (d) $\frac{35}{4}$ (e) $\frac{15\sqrt{2}}{2}$

23. The supply and demand of a certain product are each linear functions of the price charged. When the price is \$3.50, the demand is 8,000 per week, but the supply is only 6,000. When the price is \$5.00, the demand is only 3,500 per week, but the supply is 7,500.

Approximately what is the equilibrium point, i.e., the price and quantity where supply equals demand?

(a) \$5.50; 4300 items (b) \$4.67; 4500 items (c) \$4.50; 5000 items (d) \$4.00; 6500 items (e) \$3.00; 9000 items

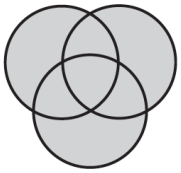
24. A pole is supported by two sets of guy wires fastened to the ground 15 meters from the pole. The shorter set of wires has slope $\pm\frac{4}{3}$. The wires in the longer set are each 50 meters long. (*The figure is not to scale!*)



Find (to the *nearest TENTH OF A DEGREE*) the measure of angle $\angle PQR$.

(a) 18.7° (b) 19.4° (c) 20.2° (d) 20.5° (e) 21.3°

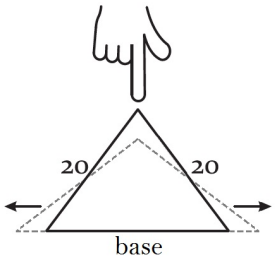
25. The circumference of each circle below passes through the centers of the other two, and the radius of each circle is 1.



What is the total gray area (that is, the area of the union of all three circle interiors)?

(a) $\sqrt{3}+\frac{7\pi}{2}$ (b) $\sqrt{3}+\frac{5\pi}{2}$ (c) $\sqrt{3}+2\pi$ (d) $\sqrt{3}+\frac{3\pi}{2}$ (e) $\sqrt{3}+\frac{3\pi}{7}$

26. The isosceles triangle below lies on a flat surface and is pushed at the top vertex. The lengths of the congruent sides do not change, but the angle between the two congruent sides will increase, and the base will stretch. Initially, the area of the triangle will increase, but eventually the area will decrease, continuing until the triangle collapses.



What is the maximum area achieved during this process?

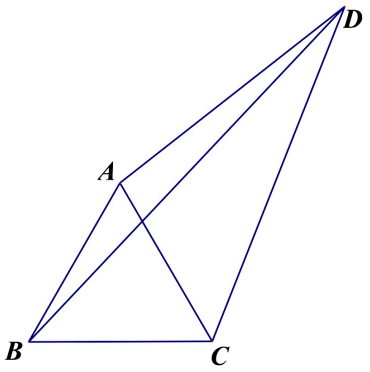
(a) 192 (b) 200 (c) 206 (d) 212 (e) 220

27. Find the value of x such that

$$11^{102} = 11 + \sum_{i=0}^{100} \log_{10}\left[(x+1)^{(11^i)}\right].$$

(a) $11^{110}-1$ (b) $11^{102}-1$ (c) $11^{101}-1$ (d) $10^{110}-1$ (e) $10^{101}-1$

28. In the figure below, quadrilateral $ABCD$ has two diagonals, AC and BD; and $\triangle ABC$ is an equilateral triangle. (*The figure is not to scale!*)



Suppose that the length of AD is 3, the length of BD is 5, and $m\angle ADC = 30°$.

What is the length of CD?

(a) $2\sqrt{3}$ (b) 4 (c) $3\sqrt{2}$ (d) $2\sqrt{5}$ (e) 4.5

29. In the equation $(\mathtt{A\,C}) \cdot (\mathtt{B\,C}) = \mathtt{D\,D\,D}$, each of the letters represents a different digit in base 10 (from 0 to 9).

What is the sum $\mathtt{A} + \mathtt{B} + \mathtt{C} + \mathtt{D}$?

(a) 19 (b) 20 (c) 21 (d) 22 (e) 24

30. How many ways can 3 Martians and 5 humans be arranged *in a circle*, if no two Martians are allowed to stand together?

(a) 960 (b) 1440 (c) 1800 (d) 3780 (e) 5040

31. On the island of New Smullyania, every inhabitant is either a *week*DAY *truthteller* (telling the truth Monday through Friday but lying on Saturday and Sunday) or a *week*END *truthteller* (telling the truth Saturday and Sunday but lying the rest of the week). At noon one day, you meet two inhabitants of New Smullyania.

Anastasia quickly tells you, "Boniface is a weekDAY truthteller."

Boniface immediately says to her, "You said that on Tuesday."

Which of the following *CANNOT* be the case?

(a) Anastasia is a weekDAY truthteller, and today is a weekDAY.

(b) Anastasia is a weekDAY truthteller, and today is on a weekEND.

(c) Anastasia is a weekEND truthteller, and today is a weekDAY.

(d) Anastasia is a weekEND truthteller, and today is on a weekEND.

(e) *Any* of the four situations listed above *could* be the case.