2013 SCSU MATH CONTEST
11th and 12th GRADE

DIRECTIONS: Select the <u>BEST</u> completion or response from among those given. Scientific and graphing calculators are allowed. Symbolic calculators are not allowed.

1. The Very Hungry Caterpillar is going to eat one apple, one pear, one plum, one strawberry, one orange and one papaya. How many different ways can he arrange his six-course meal so that the apple is eaten sometime before the orange?

   (a) 15   (b) 60   (c) 120   (d) 180   (e) 360

2. You bicycle the same route to and from school. Going to school, you pedal uphill at a constant speed of 10 MPH. Coming home from school, you cruise downhill at a constant speed of 30 MPH. What is your *average* speed in MPH for the round trip (to and from school)?

   (a) 12   (b) 15   (c) 17.5   (d) 20   (e) 22.5

3. Last month, Darth Vader ordered that a communication cable be strung tightly around the surface of the Death Star at its equator. (The Death Star is a perfect sphere of diameter 180 kilometers.) Now that the cable has been completed, he has decreed that the cable must be raised 2 meters above the surface, so that personnel may walk under it. Approximately how much *more* cable length has to be spliced into the existing cable in order to raise it everywhere to a height of 2 meters?

   (a) 6.3 m   (b) 12.6 m   (c) 56.6 m   (d) 360 m   (e) 1.130 km

4. Point P has coordinates $(3,2)$. Point Q is the reflection of P across the x-axis. Point R is the reflection of Q across the line $y = -x$. Point S is a $180°$ rotation of R around the origin. What is the area of quadrilateral $PQRS$?

   (a) 14   (b) 15   (c) 16   (d) 17   (e) 18

5. In a circle, a chord is the perpendicular bisector of a radius (which has length 12). What is the length of this chord?

   (a) $3\sqrt{3}$ (b) 27   (c) $6\sqrt{3}$ (d) $8\sqrt{3}$ (e) $12\sqrt{3}$

6. Suppose that $F(1) = 2$ and $F(n+1) = \dfrac{2F(n)+1}{2}$ for $n = 1, 2, 3, \ldots$. Determine the value of $F(101)$.

   (a) 49   (b) 50   (c) 51   (d) 52   (e) 53

7. Austin and Ally are joining four friends at a *round* table for six. How many of the possible arrangements have Austin and Ally sitting next to each other?

   (a) 15   (b) 24   (c) 48   (d) 240   (e) 288

8. The area of a rectangle remains unchanged when it is made $5/2$ inches longer and $2/3$ inches narrower, or when it is made $5/2$ inches shorter and $4/3$ inches wider. What is the area of the original rectangle?

   (a) 20 sq. in.   (b) $\dfrac{45}{2}$ sq. in.   (c) 24 sq. in.   (d) $\dfrac{80}{3}$ sq. in.   (e) 30 sq. in.

9. One thousand unit cubes are put together to form a large cube with edge length 10 units. This large cube is painted and then separated into the original cubes. How many of these cubes have at least one face painted?

   (a) 400   (b) 480   (c) 488   (d) 520   (e) 600

10. Every inhabitant of the island of Smullyania is one of two types: either a Truthteller (who *always* tells the truth) or a Liar (who *always* lies). You meet two inhabitants of the island: Yolanda and Zander.

    Zander says, "Yolanda is a Liar."

    Yolanda says, "Well, at least one of us is a Liar!"

    Which of the following is correct?

    (a) Both Yolanda and Zander are Truthtellers.

    (b) Both Yolanda and Zander are Liars.

    (c) Yolanda is a Truthteller, and Zander is a Liar.

    (d) Yolanda is a Liar, and Zander is a Truthteller.

    (e) Yolanda and Zander are of different types, but it is impossible to tell which is which.

11. At the local café, back when I was a kid, the bill for three hamburgers, seven cups of coffee, and one piece of pie was $6.50; and the bill for four hamburgers, ten cups of coffee, and one piece of pie was $8.60. What was the bill for one hamburger, one cup of coffee, and one piece of pie?

(a) $2.30   (b) $2.60   (c) $3.50   (d) $3.60   (e) not enough information is given

12. If one root of the quadratic equation $x^2 + bx + c = 0$ is $x = 2 - i$ (where b and c are real), then what is the value of c?

(a) 2   (b) 3   (c) 5   (d) 7   (e) 11

13. A class has three girls (Ann, Beth, and Carol) and three boys (Don, Ed, and Frank). All of the students line up at random. What is the probability that no boy is next to another boy, and no girl is next to another girl?

(a) $\dfrac{1}{20}$ (b) $\dfrac{1}{12}$ (c) $\dfrac{1}{10}$ (d) $\dfrac{3}{10}$ (e) $\dfrac{1}{2}$

14. A cube is inscribed in a sphere of diameter 9 inches. What is the volume of the cube, in cubic inches?

(a) 27   (b) $27\sqrt{3}$ (c) 81   (d) $81\sqrt{3}$ (e) 729

15. For positive real numbers x and y, define $x \otimes y = \dfrac{x \cdot y}{x + y}$. Which of the following statements is true?

(a) $\otimes$ is commutative, but not associative.

(b) $\otimes$ is associative, but not commutative.

(c) $\otimes$ is neither associative nor commutative.

(d) $\otimes$ is associative and commutative.

(e) None of these is true.

16. A bag contains 11 candy bars. Three cost 50 cents each, four cost $1 each, and four cost $2 each. How many ways can 3 candy bars be selected from these eleven, so that the total cost is *more than* $4?

(a) 8   (b) 28   (c) 46   (d) 66   (e) 70

17. Let (a, b) and (c, d) be two points on the line whose equation is $y = mx + k$. Which of the following represents the distance between (a, b) and (c, d)?

(a) $|a - c|\sqrt{1 + m^2}$ (b) $|a + c|\sqrt{1 + m^2}$ (c) $\dfrac{|a - c|}{\sqrt{1 + m^2}}$ (d) $|a - c|(1 + m^2)$ (e) $|a - c||m|$

18. A solid cube is sawed into eight smaller, congruent cubes. The smaller cubes are then reassembled to form the longest possible rectangular prism. Find the percentage change in the *surface area*, to the nearest *TENTH* of a percentage point.

(a) 49.8% *less*   (b) 24.3% *less*   (c) 29.4% *more*   (d) 35.5% *more*   (e) 41.7% *more*

19. Cordelia plans to apply to three Ivy League universities, four non-Ivy-League out-of-state universities, and two non-Ivy-League in-state universities. She figures that she has a 1% chance at each individual Ivy League school, a 30% chance at each non-Ivy-League out-of-state school, and a 60% chance at each non-Ivy-League in-state school. To *FOUR* decimal places, what is the probability that she will be admitted to *at least one* of the universities to which she applies? (Assume that the nine admissions decisions are independent.)

(a) 0.2772   (b) 0.4115   (c) 0.9100   (d) 0.9627   (e) 0.9982

20. If $a \neq b$, $a^3 - b^3 = 19x^3$, and $a - b = x$, then which of the following conclusions is most correct about the value(s) of a?

(a) $a = 3x$ or $a = 2x$ (b) $a = 3x$ or $a = -2x$ (c) $a = -3x$ or $a = 2x$ (d) a must be $3x$ (e) a must be $2x$

21. A cowgirl and her horse are located 4 miles south of a stream that flows due east. She is also 7 miles north and 8 miles west of her home. She wishes to water her horse at the stream and then return home. In miles, what is the shortest distance she can travel to accomplish this task?

(a) $\sqrt{32} + \sqrt{137}$ (b) $\sqrt{185}$ (c) 17   (d) $4 + \sqrt{185}$ (e) 18

22. Let P and Q be real numbers such that the fourth-degree polynomial $5x^4 + 4x^3 + 3x^2 + Px + Q$ divided by the quadratic polynomial $x^2 + 1$ has a remainder of 0. What is the value of $P - Q$?

(a) -2 (b) 0   (c) 1   (d) 2   (e) 6

23. Divide, and write your answer in the form $a + bi$ (where $i = \sqrt{-1}$): $\dfrac{2 - 3i}{4 + 5i}$

(a) $-\dfrac{7}{41} - \dfrac{22}{41}i$ (b) $-\dfrac{7}{41} + \dfrac{22}{41}i$ (c) $\dfrac{23}{41} - \dfrac{2}{41}i$ (d) $\dfrac{23}{41} + \dfrac{2}{41}i$ (e) $\dfrac{23}{41} + \dfrac{34}{41}i$

24. The two circles in the figure below are concentric (have the same center). The radius of the large circle is 10 and that of the small circle is 6. Points A and B are located on the large circle and chord AB is tangent to the small circle. What is the length of the chord AB?

(a) 8   (b) 12   (c) $12\sqrt{2}$ (d) 16   (e) $16\sqrt{2}$
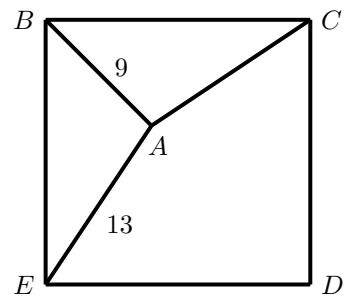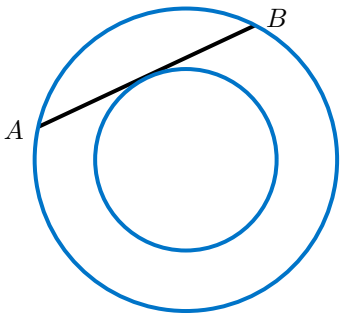
25. Point A is inside the square $BCDE$ whose area is 400 (see figure above). The length of segment AB is 9 and the length of segment AE is 13. To three decimal places, what is the length of segment AC?

(a) 16.977   (b) 17.361   (c) 17.761   (d) 18.183   (e) 18.341

26. What is the coefficient of x^5y^2 in the expansion of $(x + y + 2)^{10}$?

(a) 1008   (b) 1890   (c) 2520   (d) 20160   (e) 322560

27. A builder needs to determine the height of a house from a nearby hill. Using surveying equipment, she measures the angle of elevation ($18°$), the angle of depression ($15°$), and the horizontal distance (49 feet) from her position to the house (*below*). Determine the height of the house (in feet), to the nearest *TENTH* of a foot.

(a) 26.0   (b) 26.7   (c) 27.8   (d) 29.1   (e) 31.8
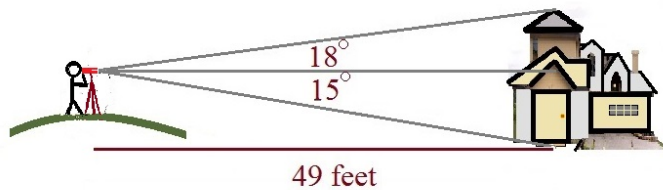
28. A positive integer is represented as 223 in base b and as 120 in base $b + 2$. What is the value of b?

(a) 5   (b) 6   (c) 7   (d) 8   (e) not enough information given

29. Adding three binary digits (i.e., each digit is 0 or 1) yields a *sum digit* and a *carry digit*. For instance, `1 + 0 + 1 = 10`, with `0` the sum digit and `1` the carry digit. It can be shown that if the binary values are stored in the Boolean variables `x`, `y`, and `z`, then the *sum* digit is generated by the following boolean expression:

`(x AND y AND z) OR (x AND NOT(y) AND NOT(z)) OR (NOT(x) AND y AND NOT(z)) OR (NOT(x) and NOT(y) AND z)`

Which of the following is a valid boolean expression for the *carry* digit?

(a) `(x AND y) OR (NOT(x) AND y AND z)`

(b) `(x AND y) OR (x AND z) OR (y AND z)`

(c) `(x AND y) AND (x OR z) AND (y OR z)`

(d) `(x OR y) AND (x OR NOT(y) OR z)`

(e) All of the above. They are logically equivalent.

30. A formal grammar for a language is defined by a start symbol, S, and a set of production rules. Consider the following set of production rules:

S *can be replaced by* T *or by* TRR.
Any T *can be replaced by* $ttTp$ *or by* tp.
Any R *can be replaced by* $qRrr$ *or by* $qqrr$.

Which of the following strings could be produced in this grammar?

(a) *tpqqrrqqrrtp*   (b) *ttpqqqrrrrqqrr*   (c) *tpqrrqqrr*   (d) *tttppqqqrrrrqqrr*   (e) *qqrrqqrr*

31. Right triangle $\triangle ABC$ has a right angle at C. Let M be the midpoint of $\overline{AB}$, and suppose that $BC = 5$ and $MC = 6.5$. To four decimal places, what is the sine of $\angle CMB$?

(a) 0.3550   (b) 0.3846   (c) 0.7101   (d) 0.7160   (e) 0.9231

32. Let A, C, and D be points on a circle O, so that $\angle CAD = 120°$, $AC = 26$, and $AD = 92$. What is the *diameter* of circle O?

(a) 62   (b) 93   (c) 124   (d) $93\sqrt{3}$ (e) $124\sqrt{3}$

33. Suppose that the length of each edge of each face of a regular icosahedron is 1 cm. What is the *surface area* (in cm^2) of the icosahedron?

(a) $5\sqrt{3}$ (b) $6\sqrt{3}$ (c) $6\sqrt{5}$ (d) $10\sqrt{3}$ (e) $12\sqrt{5}$