2011 SCSU MATH CONTEST
11^{th} and 12^{th} GRADE

DIRECTIONS: Select the <u>BEST</u> completion or response from among those given. Scientific and graphing calculators are allowed. Symbolic calculators are not allowed.

1. Sven builds 3-legged stools and 4-legged tables. Last month, Sven used 72 legs to build 3 more stools than he built tables. This month, he would like to build half as many stools as last month, but twice as many tables as last month.

   How many legs will he use this month?

   (a) 90 (b) 101 (c) 109 (d) 115 (e) 128

2. A certain natural number has base-b representation 24_b. The same number's *square* has base-b representation 554_b.

   What is b (written in base 10)?

   (a) 6 (b) 8 (c) 12 (d) 14 (e) 16

3. There are six math books, three novels, and seven history books on a bookshelf. In how many different ways can you choose four books to take on a trip, so that there is *at least one* math book among the four?

   (a) 378 (b) 720 (c) 1044 (d) 1610 (e) 1820

4. Suppose that $x < 0$. Which of the following is equal to $| 2x - \sqrt{(x-2)^2} |$?

   (a) $x - 2$ (b) $x + 2$ (c) $3x - 2$ (d) $3x + 2$ (e) $-3x + 2$

5. A kayaker paddles upriver, then back to her starting point. In still water, her paddling speed would be 5.5 MPH, but the river pushes her (whether with or against her) at 3.5 MPH. Her round trip takes exactly six hours.

   To the nearest whole mile, how many miles does she travel?

   (a) 12 (b) 20 (c) 27 (d) 33 (e) 54

6. Tweedledee invests in a fund that earns 10% the first year, 30% the second year, and 20% the third year. Tweedledum invests the same amount of money in a different fund that earns a fixed rate of return for three years. Neither Tweedledee nor Tweedledum makes any withdrawals, and at the end of the three years, they both have exactly the same amount of money in their funds.

   What was the fixed rate earned by Tweedledum?

   (a) 19.44% (b) 19.72% (c) 20.00% (d) 20.18% (e) 20.66%

7. Let the function g be given by $g(x) = ax^5 + bx^3 + cx + 9$, where a, b, and c are constants. Suppose that $g(6) = 17$.

   Which of the following is equal to $g(-6)$?

   (a) $-\frac{3}{2}$ (b) -1 (c) 1 (d) $\frac{3}{2}$ (e) 2

8. A circle is inscribed in an equilateral triangle. The circle's area is 3π.

   What is the perimeter of the triangle?

   (a) 6 (b) $6\sqrt{3}$ (c) 18 (d) $18\sqrt{3}$ (e) 36

9. Suppose you randomly permute the letters in `CONTEST`.

   What is the probability (rounded to four decimal places) that the two `T`s are *NOT* next to each other?

   (a) 0.6986 (b) 0.7143 (c) 0.8333 (d) 0.8571 (e) 0.9762

10. Suppose that f is a function with the following special property:

    For all positive real numbers a and b, $f(a \cdot b) = f(a) + f(b)$.

    Suppose also that $f(16) = 3$. What is the value of $f(2)$?

    (a) -11 (b) $\frac{3}{8}$ (c) $\frac{3}{4}$ (d) $\sqrt[4]{3}$ (e) $\sqrt{2}$

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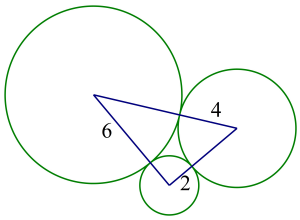
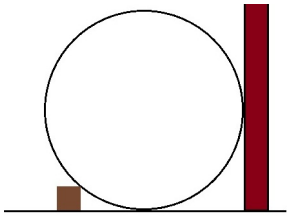
11. Three circles are tangent to each other. As indicated in the figure below left, their radii are 2 units, 4 units, and 6 units.

    What is the area of the triangle that connects their centers?

    (a) 16 (b) 24 (c) 32 (d) 48 (e) 60





12. A cylinder is pinned against a wall by a $\frac{3}{4}$-inch-high block as suggested in the figure above right. (The figure is not to scale.) The right side of the block is 5 inches away from the wall.

    To the nearest tenth of an inch, what is the radius of the cylinder?

    (a) 3.0 (b) 3.1 (c) 3.2 (d) 3.3 (e) 3.4

13. Point $A = (3, 2)$ is rotated clockwise 90° around the origin to a point A'. Then the point A' is reflected across the line $y = -1$ to a point A''.

    What is the distance between A and A''?

    (a) $\sqrt{2}$ (b) $\sqrt{26}$ (c) $5\sqrt{2}$ (d) 8 (e) $\sqrt{89}$

14. On a circular track, two people start walking from the same point P at the same time, but in opposite directions. Their speeds are 5 feet per second and 9 feet per second. They finish when they first meet again at the point P.

    How many times do they meet on the track at points other than P—that is, excluding the start and the finish?

    (a) 13 (b) 25 (c) 44 (d) 57 (e) 69

15. Every inhabitant of the island of Smullyania is either a Truthteller (and hence *always* tells the truth) or a Liar (and hence *always* lies). You meet two inhabitants of the island, Adelaide and Bernard.

    Adelaide smiles mysteriously and says, "Bernard is a Liar."

    Bernard cocks an eyebrow and says, "Yesterday Adelaide said I was a Truthteller!"

    What can you conclude?

    (a) Both Adelaide and Bernard are Liars.
    (b) Both Adelaide and Bernard are Truthtellers.
    (c) Adelaide is a Liar, and Bernard is a Truthteller.
    (d) Adelaide is a Truthteller, and Bernard is a Liar.
    (e) At least one of them is a Liar, but it is impossible to determine who.

16. You draw a card from a well-shuffled standard deck, replace the card, and re-shuffle the deck. You repeat this process until you have drawn two *consecutive* hearts.

    To four decimal places, what is the probability that you need to draw exactly 5 cards to get those two consecutive hearts?

    (a) 0.0264 (b) 0.0439 (c) 0.0625 (d) 0.2002 (e) 0.2414

17. Two farmers agree that pigs are worth \$360 and that goats are worth \$150. When one farmer owes the other money, he pays the debt in pigs or goats, with "change" received in the form of goats or pigs as necessary. For example, a \$570 debt could be paid with two pigs, with one goat received in change.

    What is the smallest debt that can be resolved in this way?

    (a) \$10 (b) \$30 (c) \$60 (d) \$90 (e) \$150

18. In $\triangle ABC$, the length of $\overline{AC}$ is 2.51, the length of $\overline{BC}$ is 4.12, and the measure of $\angle ACB$ is 107.3°.

    To the nearest hundredth, what is the length of $\overline{AB}$?

    (a) 4.82 (b) 4.95 (c) 5.19 (d) 5.24 (e) 5.42

19. My favorite quadratic function has the form $2x^2 + bx + c$. (Here b and c are real numbers that I'm not telling you.) One root of the function is $2 - i$.

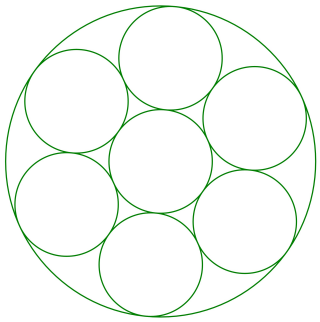
What is the value of the constant term c?

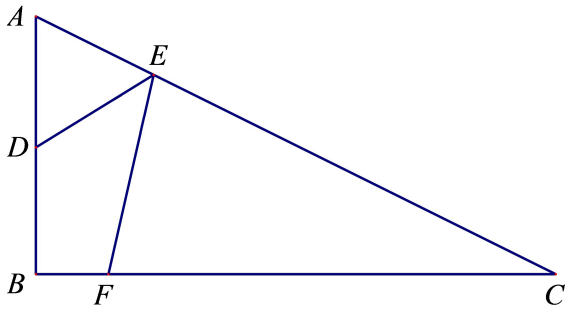
(a) 3 (b) 5 (c) 8 (d) 10 (e) 13

20. All seven small circles and the one large circle shown in the diagram below left are tangent in the manner depicted.

What is the ratio of the area of the large circle to the *combined* area of the small circles?

(a) $\frac{7}{6}$ (b) $\frac{7\pi}{18}$ (c) $\frac{8}{7}$ (d) $\frac{7}{2\pi}$ (e) $\frac{9}{7}$





21. In $\triangle ABC$ illustrated above right, $\angle ABC$ is a right angle; the lengths AD and AE are equal; and the lengths CE and CF are equal. (The figure is not meant to be exact.)

What is the measure of $\angle DEF$?

(a) 37.5° (b) 40° (c) 42.5° (d) 45° (e) 47.5°

22. What is the sum of all the solutions of $x = |3x - |35 - 3x||$?

(a) 12 (b) 35 (c) 42 (d) 47 (e) 82

23. The angle x, measured in radians, satisfies $\frac{\pi}{2} \le x \le \pi$; the angle y, measured in radians, satisfies $\pi \le y \le \frac{3\pi}{2}$.

Moreover, $\sin x = \frac{5}{13}$ and $\cos y = -\frac{3}{5}$.

Which of the following is correct?

(a) $\sin(x+y) = 33/65$, and $\cos(x+y) = 56/65$
(b) $\sin(x+y) = 33/65$, and $\cos(x+y) = -56/65$
(c) $\sin(x+y) = 56/65$, and $\cos(x+y) = 33/65$
(d) $\sin(x+y) = 56/65$, and $\cos(x+y) = -33/65$
(e) $\sin(x+y) = -56/65$, and $\cos(x+y) = -33/65$

24. A circular grass plot 12 feet in diameter is cut by a straight gravel path 3 feet wide, one edge of which passes through the center of the plot.

How many square feet are in the remaining grass area?

(a) $30\pi - 9\sqrt{3}$ (b) $36\pi - 34$ (c) $30\pi - 15$ (d) $36\pi - 33$ (e) $35\pi - 9\sqrt{3}$

25. The Jones family has five children. You know that *at least two* of the children are girls, and you assume that boys and girls are equally likely to be born.

Given this information, what is the probability that the Jones family has *at least four* girls?

(a) $\frac{1}{3}$ (b) $\frac{3}{13}$ (c) $\frac{4}{19}$ (d) $\frac{1}{2}$ (e) $\frac{4}{5}$

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26. Suppose that the function f is given by $f(x) = \dfrac{3x-2}{8+5x}$. Then the formula for the *inverse* function f^{-1} can also be written as a fraction. Which of the following is an appropriate *numerator* for $f^{-1}(x)$?

(a) $3x+5$ (b) $5x+8$ (c) $8x+2$ (d) $-2x+3$ (e) $-5x+2$

27. Professor Z. Q. Hasenpfeffer claims to have proven a wonderful theorem:

HASENPFEFFER'S THEOREM. FOR EVERY *natural number* n, IF n *has property* $\mathcal{P}$, THEN n *has property* $\mathcal{Q}$.

But the Professor's colleagues don't believe that the theorem is correct! Which of the following is logically equivalent to saying "Hasenpfeffer's Theorem is `false`"?

(a) "FOR EVERY natural number n, IF n has property $\mathcal{P}$, THEN n does NOT have property $\mathcal{Q}$."
(b) "FOR EVERY natural number n, n does NOT have property $\mathcal{P}$ AND n does NOT have property $\mathcal{Q}$."
(c) "FOR EVERY natural number n, n has property $\mathcal{P}$ BUT n does NOT have property $\mathcal{Q}$."
(d) "THERE EXISTS a natural number n such that n does NOT have property $\mathcal{P}$ AND n does NOT have property $\mathcal{Q}$."
(e) "THERE EXISTS a natural number n such that n has property $\mathcal{P}$ BUT n does NOT have property $\mathcal{Q}$."

28. Suppose that θ is an acute angle and that $\sin\left(\dfrac{\theta}{2}\right) = \sqrt{\dfrac{x-1}{2x}}$.

Which of the following is equal to $\tan\theta$?

(a) x (b) $\dfrac{1}{x}$ (c) $\dfrac{\sqrt{x-1}}{x+1}$ (d) $\sqrt{x^2-1}$ (e) $\dfrac{\sqrt{x^2-1}}{x}$

29. What are the last two digits in the sum $(1!-1)+(2!-2)+(3!-3)+\cdots+(2010!-2010)+(2011!-2011)$?

(a) 00 (b) 11 (c) 13 (d) 34 (e) 47

30. Cornelius, a math history buff, wants to try to approximate π similarly to how Archimedes did it two millennia ago. He finds the average of the perimeters of two regular octagons (eight-sided figures), one circumscribing a circle of radius 1 and one inscribed in that circle. This average is his *estimate* of the circumference of the circle.

Based on this estimate for the circle's circumference, what value (to *four* decimal places) does Cornelius calculate for π?

(a) 3.1187 (b) 3.1493 (c) 3.1722 (d) 3.1876 (e) 3.2170

31. Consider the parabola given by the equation $y = x^2$. For any nonzero real number u, let l_u be the normal line to this parabola (that is, the line perpendicular to the tangent line) at the point (u, u^2). Let $(0, b_u)$ be the y-intercept of l_u.

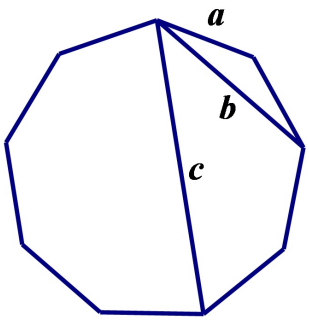
Then as u approaches 0, b_u approaches... *what*?

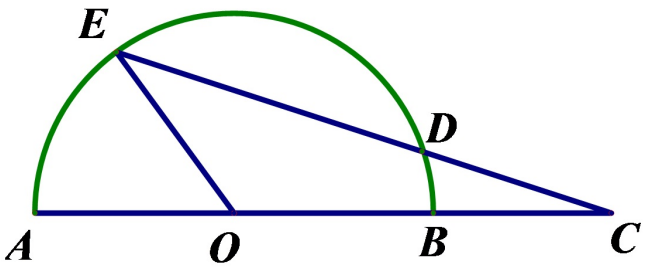
(a) 0 (b) $\dfrac{1}{\sqrt{2}}$ (c) $\dfrac{1}{2}$ (d) 1 (e) ∞

32. Suppose that a, b, and c are the lengths of a side, a shortest diagonal, and a longest diagonal, respectively, of a regular nonagon (see the figure below left).

Which of the following relationships is correct?

(a) $c = a+b$ (b) $c^2 = a^2+b^2$ (c) $c^2 = a^2+ab+b^2$ (d) $b = \dfrac{a+c}{2}$ (e) $b = \sqrt{ac}$





33. In the figure above right, $\overline{AB}$ is the diameter of the semicircle with center O, the lengths CD and OA are equal, and the measure of $\angle AOE$ is 45°. (The figure is not meant to be exact).

What is the measure of $\angle BCD$?

(a) 10° (b) 15° (c) 20° (d) 25° (e) 30°