

2004 SCSU MATH CONTEST
11th and 12th Grade Test

Please note: Solutions may be obtained in a variety of ways, many of which make ingenious use of the calculator. The solutions suggested here are intended as hints for ways of thinking about problems, not as required methods. Hopefully they will inspire students to consider ramifications or experimental changes in the problems, because the objective of the contest is to inspire creative thought.

Scientific and graphic calculators are allowed. Symbolic calculators are not allowed.

Directions: Select the one best response for each of the following problems.

1. If $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, then $\frac{2}{x}$ equals

- (a) -1 (b) 1 (c) 2 (d) -1 or 2 (e) -1 or -2.

Solution: $1 - \frac{4}{x} + \frac{4}{x^2} = 0$, so $\left(1 - \frac{2}{x}\right)^2 = 0$ or $\frac{2}{x} = 1$.

2. Judith travels m feet north at 2 minutes per km. She returns due south to her starting point at 2 km per minute. The average rate in km per minute for the entire trip is:

- (a) 2/3 (b) 1/2 (c) 4/5 (d) 3 (e) none of these

Solution: The length of the trip is could just as well be 1 unit. Judith's rate going north is $\frac{1}{2}$ km/ min.

Her time north is thus $1 / (1/2) = 2$, her time south is $1/2$. Rate = distance/time = $\frac{2}{\frac{1}{2} + 2} = \frac{2}{5/2} = \frac{4}{5}$

3. Find the area of the region of the plane bounded above by the graph of $x^2 + y^2 + 6x = -8$ and below by the graph of $y = |x + 3|$.

- (a) $\pi/4$ (b) $\pi^2/4$ (c) $\pi/2$ (d) π (e) $2\pi^2$

Solution: Completing the square, the top curve is the circle $(x + 3)^2 + y^2 = 1$ and the lower curve which is a right angle "vee-shape", cuts the circle (of radius 1) into quarters. The area of the region is

$\frac{1}{4}\pi \cdot 1^2 = \frac{\pi}{4}$

4. If the statement "All iguanas in this store are green" is false, which of the following statements *must* be true?

- I. All iguanas in this store are other than green.
 II. There is some iguana in this store that is not green.
 III. No iguana in this store is green.

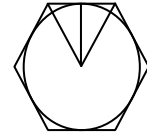
IV. Not all iguanas in this store are green.

- (a) II only (b) IV only (c) I and III only (d) II and IV only (e) I, II and IV only

Solution: Only II and IV are true.

5. The area of a circle inscribed in a regular hexagon is 100π . The area of the hexagon is:

- (a) $120\sqrt{5}$ (b) $200\sqrt{2}$ (c) 300 (d) $200\sqrt{3}$ (e) 600



Solution: The radius of the circle is 10. The side of the hexagon, s , satisfies the equation.

$\frac{\sqrt{3}}{2}s = 10$; namely $s = \frac{20}{\sqrt{3}}$. The area of one of the equilateral triangles is

$Area = \frac{1}{2} \text{base} * \text{height} = \frac{10}{2} \left(\frac{20}{\sqrt{3}} \right)$. This number is multiplied by 6 for the 6 equilateral triangles, and the

answer to the area of the hexagon is $Area = 200\sqrt{3}$

6. The repeating decimal $0.810810810\dots$ can be written as a fraction. When the fraction is written in lowest terms, what is the *sum* of the numerator and the denominator?

- (a) 20 (b) 67 (c) 109 (d) 144 (e) 191

Solution: The decimal is $\frac{810}{999} = \frac{30}{37}$ and numerator + denominator = 67

7. Consider the integers 1, 2, 3, \dots , 100. How many of these are divisible by neither 5 nor 7?

- (a) 52 (b) 60 (c) 65 (d) 68 (e) 72

Solution: There are 100 integers, and 20 are divisible by 5 and denoting by $[[x]]$ the “floor” or “greatest integer less than or equal to” function, the number of numbers divisible by 7 is $[[100/7]] = 14$. This overcounts those integers which are divisible by 35—there are two such, since $[[100/35]] = 2$. Thus the number of integers divisible by 5 **or** 7 (allowing division by 5 or 7) in this set is $20 + 14 - 2 = 32$. The number **not** divisible by these numbers is $100 - 32 = 68$.

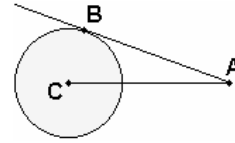
8. A corporation has 10,000 computers, each labeled with a unique number from 1 to 10,000. What is the probability that the number of a computer chosen at random contains no 7's?

- (a) $(9/10)^4$ (b) $(7/10)^2$ (c) $9/10$ (d) $(9/10)^2$ (e) $(7/10)^4$

Solution: In each position each digit is as likely as any other, and the numbers that appear in the various positions occur independently. Thus the probability that 7 does not appear in one particular position is $9/10$ and the probability that a number contains no 7's is $(9/10)^4$.

9. The radius of the circle is 1, and the distance from C to A is 3. How far is B from A?

- (a) 2 (b) $2\sqrt{2}$ (c) $2\sqrt{3}$ (d) $\sqrt{10}$ (e) $3\sqrt{2}$



Solution: Draw the radius BC (which makes a right angle with the tangent line AB) and let x be the length of AB then $x^2 + 1 = 3^2$ so $x = \sqrt{8} = 2\sqrt{2}$.

10. If $\log_k x \cdot \log_5 k = 3$ and k is positive, then x equals:

- (a) k^3 (b) k^5 (c) $5k^3$ (d) 125 (e) 243

Solution: Let $a = \log_k x$ and $b = \log_5 k$. Then $k^a = x$ and $5^b = k$. Thus $x = k^a = (5^b)^a = 5^{ab} = 5^3 = 125$

11. The lines L and K are symmetric to each other with respect to the line $y = x$. If the equation of the line L is $y = ax + b$ with a and b not zero, then the equation of K is $y = ?$

- (a) $\frac{x}{a} - \frac{b}{a}$ (b) $-\frac{x}{a} - b$ (c) $-\frac{x}{a} + \frac{b}{a}$ (d) $\frac{x}{a} + \frac{b}{a}$ (e) $-\frac{x}{a} - \frac{b}{a}$

Solution: What is being asked for is the inverse of the function $f(x) = ax + b$. Namely if $y = ax + b$, we are to solve for x in terms of y : namely, $x = \frac{y - b}{a}$. The reflection about the line $y = x$ creates the need to swap x and y in the equation directly above. Thus $y = \frac{x - b}{a} = \frac{x}{a} - \frac{b}{a}$.

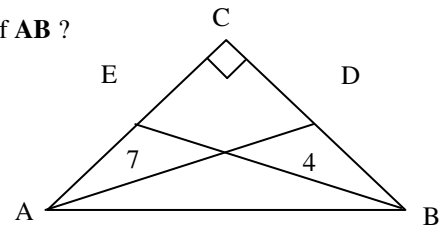
12. If $xy = b$ and $\frac{1}{x^2} + \frac{1}{y^2} = a$, then $(x + y)^2$ equals:

- (a) $(a + 2b)^2$ (b) $a^2 + b^2$ (c) $b(ab + 2)$ (d) $ab(b + 2)$ (e) $\frac{1}{a} + 2b$

Solution: (d). Since $a = \frac{1}{x^2} + \frac{1}{y^2} = \frac{x^2 + y^2}{(xy)^2} = \frac{x^2 + y^2}{b^2}$, so $(x + y)^2 = x^2 + 2xy + y^2 = x^2 + y^2 + 2xy = ab^2 + 2b = b(ab + 2)$.

13. AB is the hypotenuse of a right triangle ABC . The length of median AD is 7, and the length of median BE is 4. What is the length of AB ? (The graph is not to scale.)

- (a) $5\sqrt{3}$ (b) $5\sqrt{2}$ (c) 3 (d) $2\sqrt{5}$ (e) $2\sqrt{13}$



Solution: Let a be the length of side BC and b the length of side AC, while c is the length of AB. Now

$$a^2 + \left(\frac{b}{2}\right)^2 = 4^2 \text{ and } b^2 + \left(\frac{a}{2}\right)^2 = 7^2 \text{ so adding these equations together, we have } \frac{5}{4}a^2 + \frac{5}{4}b^2 = 65.$$

Thus $c^2 = a^2 + b^2 = 52$ giving $c = 2\sqrt{13}$.

14. How many distinguishable 6-letter sequences can be made from the letters in SEESAW? (You must use each letter exactly as many times as it appears in SEESAW.)

- (a) 24 (b) 180 (c) 240 (d) 360 (e) 720

Solution: Think of the word as $S_1E_1E_2S_2AW$. The number of spellings of $S_1E_1E_2S_2AW$ with distinguishable letters is $6!$. However if we spell SEESAW this way (now the S's and E's, repeated letters, are indistinguishable) we will be overcounting as follows: for every spelling of SEESAW there are 2 spellings in which the E's are indistinguishable, and 2 in which the S's are. Thus there are $6! / 4$ spellings. This number is 180.

15. An ant starts at the point $O = (0,0)$ in the xy -plane. He walks along a line that has slope 1 for one meter. He then turns and walks along a line with slope $\sqrt{3}$ for one meter and plants a flag at point F . What is the angle (in degrees) which the line OF makes with the positive x -axis?



Solution: The angle θ is $360^\circ - (45^\circ + 90^\circ + 60^\circ) = 165^\circ$. Since triangle ABC is isosceles, the angle $\theta = 15/2$. Thus the angle desired is $45^\circ + (15/2)^\circ = 52\frac{1}{2}^\circ$.

Or: Creating triangles, the first step takes the ant to $B = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, and the next step to the point

$C = \left(\frac{\sqrt{2}}{2} + \frac{1}{2}, \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2}\right)$. The tangent of this angle is the y -coordinate divided by the x -coordinate. The

calculator finds the inverse tangent to be $52\frac{1}{2}^\circ$ degrees, although this can be arrived at without calculator.

16. A problem in mathematics is given to John and Jake whose chances of solving it are $\frac{1}{2}$ and $\frac{3}{4}$, respectively. What is the probability that the problem will be solved by at least one of them?

- (a) $\frac{1}{4}$ (b) $\frac{3}{8}$ (c) $\frac{5}{8}$ (d) $\frac{7}{8}$ (e) $\frac{8}{9}$

Solution: $P(\text{John solves}) + P(\text{Jake solves}) - P(\text{John and Jake solve}) = \frac{1}{2} + \frac{3}{4} - (1/2)*(3/4) = 5/4 - 3/8 = 10/8 - 3/8 = 7/8$.

17. What is $\tan(\sin^{-1}(x))$?

- (a) x (b) $\cos x$ (c) $\sqrt{1-x^2}$ (d) $x/\sqrt{1-x^2}$ (e) $1/\sqrt{1-x^2}$

Solution: Drawing the triangle with angle theta and sides x and 1 so that $\sin(\theta) = x/1$, the remaining side has length $\sqrt{1-x^2}$. The tangent of this theta is $\frac{x}{\sqrt{1-x^2}}$.

18. How many solutions are there to the equation $|2x - 1| - |x + 5| = 3$?

- (a) 0 (b) 1 (c) 2 (d) 3 (e) 4

Solution : One can graph $y = |2x-1|$ and $y = 3 + |x+5|$ and see two intersections. Or one can create an argument using slopes.

19. Given that $i^2 = -1$, what is $(1+i)^{2004} - (1-i)^{2004}$?

- (a) 0 (b) 1 (c) i (d) 2i (e) 4 + i

Solution : This quantity is the difference of squares, namely, $((1+i)^2)^{1002} - ((1-i)^2)^{1002}$

But since $(1+i)^2 = 1+2i+i^2 = 2i$ and $(1-i)^2 = 1-2i+i^2 = -2i$, the result is

$$((1+i)^2)^{1002} - ((1-i)^2)^{1002} = (2i)^{1002} - (-2i)^{1002} = 0.$$

20. In my town, everyone is either a *knight* or a *knave*. Knights always tell the truth,

and knaves always lie. One day you meet three people. “B and I are not both knaves,” says A. “Also, A and I are not both knights,” says C. Meanwhile, B remains silent. Which of the following is true?

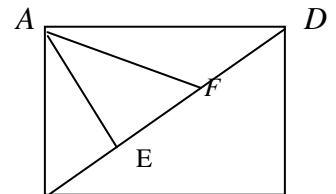
- a) B is the only knight. b) C is the only knight. c) A and C are the only knights.
d) All of these citizens are knights. e) None of these citizens is a knight.



Solution: Suppose C is a knave. Then his statement is false, so he and A ARE both knights. So C is not a knave, but a knight. Then, his statement being true, A is not a knight. Since A is a knave and his statement is false, we have that A and B are both knaves. Thus C is the only knight.

21. Rectangle ABCD has a length of 4 cm and a width of 3 cm. Diagonal BD is divided into three equal segments by points E and F. Find the area of the triangle ΔAEF in cm^2 . (Graph may not be drawn to scale.)

- (a) $\sqrt{2}$ (b) 2 (c) $\sqrt{3}$ (d) $\frac{\sqrt{3}}{2}$ (e) $\frac{2}{\sqrt{3}}$



Solution: The three small triangles have equal area; half the area of the rectangle is 6. Thus the area of a single triangle, such as triangle AEF is $6/2 = 3$.

B

C

22. What is the measure in radians of the acute angle between any two crossing long diagonals of the cube?

(a) $\tan^{-1}(1/\sqrt{2})$ (b) $\frac{\pi}{3}$ (c) $\frac{\pi}{2}$ (d) $2 \sin^{-1}(\sqrt{3}/3)$ (e) $\cos^{-1}(\sqrt{3}/2)$

Solution: The length of the edge of a cube can be one unit. The diagonal has length $\sqrt{(\sqrt{1^2 + 1^2})^2 + 1^2} = \sqrt{3}$. Therefore, half a diagonal has length $\frac{\sqrt{3}}{2}$. Half a side has length $\frac{1}{2}$. Constructing a triangle, with theta = $\frac{1}{2}$ of the acute angle at the intersection of diagonals, we find $\sin(\theta) = \frac{1}{\sqrt{3}}$. The angle we want is the double of this. Thus the angle at the intersection is $2 \sin^{-1}(\sqrt{3}/3)$.

23. The equation $2^{2x} - 8 \cdot 2^x + 12 = 0$ is satisfied by $x = 1$ and also $x =$

(a) $\log_{10} 3$ (b) $\frac{\ln 6}{2}$ (c) $1 + \frac{\ln 3}{2}$ (d) $\frac{\ln 6}{\ln 2}$ (e) 4

Solution: Letting $y = 2^x$, we have $y^2 - 8y + 12 = 0$ or $(y-6)(y-2) = 0$. Using $y = 6$, we have $y = 2^x = 6$ and $x = \ln 6 / \ln 2$. The other solution, $y = 2$, yields $y = 2^x = 2$ and $x = 1$.

24. The function f is defined on the interval $[0,1]$ by

$$f(x) = \begin{cases} x + \frac{1}{2}, & 0 \leq x < \frac{1}{2} \\ 2 - 2x, & \frac{1}{2} \leq x \leq 1 \end{cases}$$

Let f^n denote the n^{th} iteration of f , so $f(f(f \dots f(1/2))) = f^{2004}(1/2) = ?$
(2004 times)

(a) 1 (b) 0 (c) 1/2 (d) 1/4 (e) 3/4

Solution: Since $f^3(1/2) = 1/2$ and 2004 is evenly divisible by 3 the answer is

$$f(f(f \dots f(1/2))) = f^{2004}(1/2) = 1/2$$

(2004 times)

25. Define $a \oplus b = \frac{ab}{a+b}$ and $a \otimes b = \frac{1}{a} + \frac{1}{b}$, $a, b, a+b$ are nonzero real numbers.

If $2 \otimes (k \oplus 3) = 0$, then k is

(a) -7/2 (b) -3 (c) -6/5 (d) 1/3 (e) 2/3

Solution: By evaluating this, $k = -6/5$

26. If a and b are positive numbers, and $\log_{b^2} a^2 = 3$, then $\log_b a = ?$

- (a) 1 (b) 2 (c) 3 (d) 4 (e) 5

Solution: Since $(b^2)^3 = a^2$, we have $b^6 = a^2$, so $b^3 = a$ and $\log_b a = 3$

27. In how many ways can \$5.00 be made of nickels and pennies?

- (a) 50 (b) 98 (c) 99 (d) 100 (e) 101

Solution: Consider \$5.00; we can have 100 nickels and 0 pennies, or 99 nickels and a set of pennies, or 98 nickels and a set of pennies,... or 500 pennies. This listing contains 101 ways in which \$5.00 can be divided.

28. Let a_1, a_2, \dots and b_1, b_2, \dots be arithmetic progressions such that $a_1 = 25$, $b_1 = 75$, and $a_{100} + b_{100} = 100$. Find the sum $(a_1 + b_1) + (a_2 + b_2) \dots (a_{100} + b_{100})$.

- (a) 175 (b) 200 (c) 750 (d) 10,000 (e) 15,000

Solution: Since $a_1 + b_1 = 100$ and $a_{100} + b_{100} = 100$, the difference between terms in the arithmetic progression is 0, so each term of the sequence is 100. There are 100 terms. Their sum is 10,000.

29. If $f(x)$ is a fifth-degree polynomial with exactly two distinct x -intercepts, then f has:

- (a) exactly one complex root (b) exactly 2 complex roots (c) exactly 3 complex roots
(d) a graph with a vertical asymptote (e) no y -intercept

Solution: A fifth degree polynomial splits into 5 factors, with 5 roots, some possibly complex. Since there are only 2 real roots, and since complex roots occur in pairs (the complex root occurs with its conjugate as another root), one of the real roots is repeated. There are 2 complex roots, and 2 distinct real roots, with one being repeated.

30. What is the smallest angle created between the minute hand and the hour hand at 3:15 on an analog clock (assuming that the clock hands move continuously)?

- (a) 0 degrees (b) 5 degrees (c) $15/2$ degrees (d) 15 degrees (e) none of these

Solution: The hour hands moves $360/12 = 30$ degrees per hour. At 3:00 the hour hand is at 90 degrees with the minute hand at 0 degrees. Fifteen minutes later, the hour hand as moved $30/4 = 15/2$ degrees below 0 degrees, and the minute hand is at 0 degrees. The difference is $15/2$ degrees.

31. How many solutions with x, y integers, are there for the equation $3x + 6y = 136$?

- (a) 0 (b) 1 (c) 2 (d) 24 (e) infinitely many

Solution: The left hand side is divisible by 3; i.e. 3 is a factor of the left hand side, while 3 is not a factor of 136. Thus there are no solutions among integers.

32. My calculator is frustrating! It has a key which replaces the displayed entry with its square and it has another key which replaces the displayed entry with its reciprocal. I start with an entry x and alternately squaring and reciprocating, n times each, I get the result y . Then y is

- (a.) $x^{((-1)^n 2n)}$ (b.) x^{2n} (c.) x^{-2n} (d.) $x^{-(2^n)}$ (e.) $x^{((-2)^n)}$



Solution: Every reciprocal can be thought of as giving x^{-1} ; every squaring as x^2 . This happens n times so the answer is $x^{((-2)^n)}$.

33. The sequence $\{F_n\}_{n=0}^{\infty}$ of Fibonacci numbers is defined by $F_0 = 0, F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}, n = 1, 2, \dots$. What is the last digit of F_{60} ?

- (a) 0 (b) 1 (c) 2 (d) 3 (e) none of these

Solution: $F_0 = 0, F_1 = 1, F_2 = 1, F_{n+2} = F_{n+1} + F_n, n \geq 2$

$\{F_n\}_{n=0}^{\infty}$: 0 1 1 2 3 5 8 13 21 34 55 89 144 ...

$\{F_n\}_{n=0}^{\infty} \pmod{2}$: 0 1 1 0 1 1 0 1 1 0 ...

$F_{n+3} = F_n, \pmod{2} \Rightarrow F_{60} \equiv 0 \pmod{2}$

$\{F_n\}_{n=0}^{\infty} \pmod{5}$: 0 1 1 2 3 0 3 3 1 4 0 4 4 3 2 0 2 2 4 1 0 1 1 2 3 ...

$F_{n+20} = F_n, \pmod{5} \Rightarrow F_{60} \equiv 0 \pmod{5}$

$2 \mid F_{60}, 5 \mid F_{60} \Rightarrow 10 \mid F_{60}$ since $\gcd(2, 5) = 1$.

Thus $F_{60} \equiv 0 \pmod{10}$, that is, the last digit of F_{60} is 0.