Modeling Smooth Structural Changes in the Trend of US Real GDP

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Abstract

A key feature of Gallant's Flexible Fourier Form is that the essential characteristics of one or more structural breaks can be captured using a small number of low frequency components from a Fourier approximation. We introduce a variant of the Flexible Fourier Form into the trend function of U.S. real GDP in order to allow for gradual effects of unknown numbers of structural breaks occurring at unknown dates. We find that the Fourier components are significant and that there are multiple breaks in the trend. In addition to the productivity slowdown in the 1970s, our trend also captures a productivity resumption in the late 1990s and a slowdown in the late 1950s. Our cycle corresponds very closely to the NBER chronology. We compare the decomposition from our model with those from a standard unobserved components model, the HP filter, and the Perron and Wada (2005) model. We find that our decomposition has several favorable characteristics over the other models and has very different implications about the recovery from the recent recession.

JEL Classifications: E32, E37, C32

Key Words: Flexible Fourier Form, Smooth Trend Breaks, Fourier Approximation

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1. Introduction

The appropriate way to decompose U.S. real GDP into a trend and cyclical component has received a substantial amount of attention in the recent macroeconometrics literature. The most commonly used decomposition methods include the Unobserved Components model (UC), the Beveridge-Nelson decomposition (BN), and the Hodrick-Prescott (HP) filter. Beveridge and Nelson (1981) define the trend as the limiting forecast as the horizon goes to infinity, adjusted for the mean rate of growth. They show that the BN trend is a random walk plus a drift. Clark (1987) decomposed real U.S. output using an unobserved components model such that the trend is a random walk plus drift and the cyclical component is a stationary AR(2) process. A number of papers have tried to improve on Clark's specification of the cycle. For example, Harvey and Jaeger (1993) specify a model of the cycle based on a cosine wave, Kim and Nelson (1999) allow the cycle to be a nonlinear Markov switching cyclical process, and Koopman, Lee and Wong (2006) allow the parameters of the cycle to be a smooth transition process based on trigonometric functions. It is well known that the standard UC model yields very different trend-cycle decompositions than the BN method. In particular, the UC trend is smooth and the UC cycle is large and highly persistent, while the BN trend is volatile and the BN cycle is small and noisy. Recently, Morley, Nelson, and Zivot (2003, hereafter MNZ) show that once we allow for non-zero trend-cycle correlation in the UC model, the so-called UC_UR model yields the same decomposition as the BN method, i.e., very volatile trend and small and noisy cycle.

For our purposes, the extensions of the BN and UC models have focused on the cycle with relatively little effort going into the specification of the trend. However, if the trend is misspecified, the cycle is necessarily misspecified as well. A notable exception is that of Perron and Wada (2005, hereafter PW); PW provide evidence that it is important to account for a structural break in the trend of U.S. real GDP. They use a dummy variable to capture the structural break in the slope of the trend occurring in 1973:Q1. They find that the UC_UR model and BN method yield the same trend-cycle decomposition: the trend is deterministic except for a productivity slowdown occurring in 1973:Q1, and that the implied cycle corresponds very closely to the NBER chronology. As such, they claim that previous studies misspecify the trend function by ignoring the structural break. A similar argument is made by Basistha (2007) for the Canadian economy.

There are several reasons to believe that the drift term in the trend is more complicated than a simple break in the drift occurring at 1973:Q1. First, the break date (1973:Q1) is exogenously chosen in PW. However, the exact time of the productivity slowdown is not so obvious. While many studies suggest that the productivity slowdown began in the early 1970s, there are many others pointing to a decline in the productivity starting in the late 1960s (see, for instance, the Symposium on the Slowdown in productivity growth in the *Journal of Economic Perspectives*, 1988). The ambiguity of the break date calls for a model that allows for breaks occurring at unknown dates so that we do not need to pre-choose the break date. Second, by using a dummy variable, PW assume that the break manifests itself contemporaneously. In other words, they assume that the productivity slowdown was a one-time decrease in trend GDP occurring precisely at 1973:Q1. However, there are strong reasons to believe that the productivity slowdown was a gradual process instead of an abrupt decrease. A primary suspect of the cause of the productivity slowdown is the 1973 oil price increase, which was not a one-time increase but a series of increases. During 1973, OPEC increased posted prices by 5.7% on April 1, 11.9% on June 1, 17% on October 16, and declared an export embargo on October 20. Moreover, the effects of the oil price increases on real GDP are likely to be gradual as it takes time for the price increases to manifest themselves in output reductions. Other studies have suggested other possible causes of the productivity slowdown, such as the slowdown in the R&D activity and the decline in the rate of increase of the skill of the labor force during the 1970s, all of which were gradual economic phenomena. As such, their impact on productivity was more likely to be sustained and smooth. Therefore, it seems plausible to allow for some form of smooth break. Third, PW assume that there is only one break in the trend. We suspect that there exists more than one break. For example, there is evidence of a resumption of high productivity growth in the 1990s. A model that allows for unknown numbers of breaks would be much more desirable.

In this paper, we allow for gradual effects of unknown numbers of structural breaks occurring at unknown dates. Specifically, we introduce a variant of Gallant's (1981) Flexible Fourier Form (FFF) into the trend function of US real GDP. It has been well-demonstrated by Becker, Enders, and Hurn (2004) and Becker, Enders, and Lee (2006) that the essential characteristics of one or more unknown structural breaks can be captured using a small number of low frequency components from a Fourier approximation. One important feature the FFF is that we do not need to assume that the break dates or the number of breaks are known *a priori*. Instead of selecting specific break dates, the number of breaks, and the form of the breaks, the specification problem is transformed into incorporating the appropriate frequency components into the estimating equation. Note that PW also use an alternative model to capture smooth breaks: they adopt a generalized state space model where the errors have non-Gaussian distributions. However, they acknowledge that it is necessary for them to impose a number of non-standard restrictions to achieve identification. They also point out (PW, p16.) that these restrictions affect the parameter estimates significantly and that "more work is needed to carefully assess the identification of such models." By contrast, our model is identified so that we do not need to impose any other restrictions that might affect the reliability of our results. Moreover, our approach spares the computational complexity involved in the generalized state space models. The non-Gaussian feature of the generalized state space models complicates the filtering methodology substantially. Our model can be estimated directly using the basic Kalman filter.

The aim of the paper is to compare the trend and cyclical estimates of real U.S. GDP from our FFF model to those from the UC_UR model, the HP filter, and the PW model. Note that our model is identical to the MNZ model except that we include Fourier components in the trend. It is also identical to the PW model except for the fact that we use the Fourier components rather than a 1973:Q1 dummy variable in the trend. We show that our decomposition has several favorable characteristics over the alternatives. The UC_UR trend is very volatile and does not have any obvious pattern, and the UC_UR cycle is small and noisy and bears little resemblance to the NBER chronology. The HP trend completely misses the productivity slowdown in the 1970s and does not seem to capture the productivity resumption in the early 2000's. When we use the FFF to model trend GDP, we reconfirm the PW result that the estimated standard deviation of the shocks to the trend is zero. The suggestion is that the trend is deterministic once unknown breaks are accounted for. However, unlike PW, we find that there are multiple breaks in the trend: in addition to the productivity slowdown in the 1970s, our trend also captures the productivity resumption in the late 90's and early 2000's, and a productivity slowdown in the late 1950s. Moreover, our estimated cycle corresponds very closely to the NBER chronology and suggests a slower recovery from the 2001 recession than the HP filter and the PW model.

We also find that the symmetry of the cycle is substantially affected by the different trend specifications. Models with highly volatile trend, such as the MNZ model, tend to attribute most fluctuations in the GDP series to the trend and produce symmetric cycle series. On the contrary, models with very smooth trend, such as the PW model, tend to produce asymmetric cycles. Our cycle is more symmetric than the MNZ cycle but less symmetric than the PW model.

The rest of this paper is organized as follows. Section 2 provides a description of our methodology for using a Fourier series to approximate unknown structural breaks. Section 3 presents the model. Section 4 discusses the estimation results and compares the decomposition from our model to those from other models. Section 5 concludes.

2. Approximating Structural Breaks with a Fourier Series

A Fourier series approximation for the intercept in a time-series model has the following form:

$$\alpha(t) = \mu + \sum_{k=1}^{n} \alpha_k \sin(2\pi kt / T) + \sum_{k=1}^{n} \beta_k \cos(2\pi kt / T); \quad n \leq T / 2 \tag{1}$$

where n represents the number of frequencies contained in the approximation, k represents a particular frequency, and T is the number of observations. The Fourier series is capable of approximating absolutely integrable functions to any desired degree of accuracy. Beginning with $n = 1$, it is always possible to improve the approximation by using additional frequencies. When $n = T/2$ is reached, the fit of the approximation will be perfect. As shown in Becker, Enders, and Hurn (2004) and Becker, Enders, and Lee (2006), an important feature of the FFF is that essential characteristics of one or more structural breaks can be captured using a small number of low frequency components from a Fourier approximation. This is so since breaks tend to shift the spectral density function towards frequency zero. It is not necessary to assume that the break dates or the number of breaks are known *a priori* or that the breaks are sharp. In the absence of structural breaks or other forms of nonlinear trend, all values of $\alpha_k = \beta_k = 0$.

Panels 1 to 6 of Figure 1 illustrate the ability of the FFF to mimic the behavior of smooth breaks in the trend of a time series. The solid lines in the six panels show smooth transition models of trend shifts. Panel 1 shows the following smooth transition logistic (LSTAR) model of a trend break with parameter values $d_1 = 2$, $\gamma = 0.05$, $T = 500$, $\delta = 0.015$, and $\lambda = 0.5$:

$$y_t = d_1 / [1 + \exp(\gamma(t - \lambda T))] + \delta t + e_t$$

Panel 2 of the figure shows the same LSTAR break setting $\lambda = 0.75$. Panel 3 shows the following exponential break (ESTAR) using parameter values $d_1 = 2$, $\gamma = 0.0002$, $T = 500$, $\delta = 0.01$, and $\lambda = 0.5$:

$$y_t = d_1[1 - \exp(-\gamma(t - \lambda T)^2)] + \delta t + e_t$$

Panels 4 through 6 show series with multiple smooth breaks. In each panel, the short-dashed line and the long-dashed line show the time path of $y_t = \alpha(t) + \delta t$ obtained by setting n = 1 and n = 2, respectively:

$$y_t = \mu + \alpha_1 \sin(2\pi t/T) + \beta_1 \cos(2\pi t/T) + \delta t \quad (2)$$

$$y_t = \mu + \alpha_1 \sin(2\pi t/T) + \beta_1 \cos(2\pi t/T) + \alpha_2 \sin(4\pi t/T) + \beta_2 \cos(4\pi t/T) + \delta t \quad (3)$$

The values of α's, β's, and δ are obtained by regressing y_t on a constant, a linear time trend t, and the sinusoidal components.

Although these types of smooth breaks might be expected to be seen in economic variables with a slowly evolving trend, there is no simple way to incorporate such LSTAR or ESTAR trend functions in an unobserved components framework. Nevertheless, from the six panels of Figure 1, we can see that a Fourier approximation using the single frequency $n = 1$ can mimic unknown smooth breaks with fairly high degrees of accuracy. The centered R^2 from equation (2) is between 0.953 and 0.994. It is obvious that the second frequency component (n = 2) improves the fit somewhat. The point is that one or two frequency components capture much of the variation in the trend. Moreover, the approximation does not require that the pattern of the break be symmetric.

Although the Fourier series is especially suitable to mimicking smooth breaks, we also illustrate sharp breaks in Figure 2. The solid lines in panels 1 to 4 show six series for T = 100 with the types of sharp breaks used in Becker, Enders, and Lee (2006). Panel 1 shows a sharp break at $T/2$. Panel 2 shows multiple breaks. Panels 3 through 6 illustrate breaks in the slope and intercept of the trend. As in Figure 1, the short-dashed line depicts the Fourier approximation using the single frequency n = 1 and the long-dashed line depicts the approximation using two frequencies n = 2. The values of α 's, β 's, and δ are obtained by regressing the DGP on a constant, a linear time trend t, and the sinusoidal components. The six panels of Figure 2 reinforce the key points illustrated by Figure 1. A value of n = 1 or n = 2 is sufficient to capture the essential features of many types of breaks. The addition of the second frequency seems to be important if there are several breaks and/or if the breaks are sharp. Of course, it is possible to design series with more complicated break patterns than those shown in Figures 1 and 2. However, if the Fourier approximation cannot capture the nature of such breaks, the estimated values of α_k and β_k should not be statistically different from zero.

3. The Model

Our model uses the same basic set-up as MNZ and PW except that we introduce Fourier components into the trend function of U.S. real GDP:

$$y_t = \tau_t + c_t \tag{4}$$

$$\tau_t = \mu + \sum_{k=1}^{n} \alpha_k \sin(2\pi kt/T) + \sum_{k=1}^{n} \beta_k \cos(2\pi kt/T) + \tau_{t-1} + \eta_t \tag{5}$$

$$\phi_p(L)c_t = \theta_q(L)\varepsilon_t \tag{6}$$

$$\begin{bmatrix} \eta_t \\ \varepsilon_t \end{bmatrix} \sim i.i.d.N\left(\begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} \sigma_\eta^2 & \sigma_{\eta\varepsilon} \\ \sigma_{\eta\varepsilon} & \sigma_\varepsilon^2 \end{bmatrix} \right) \tag{7}$$

where y_t is log U.S. real GDP, τ_t is the unobserved trend, c_t is the unobserved stationary cycle, and $n < T/2$ is the number of frequency components contained in the model.[1]

The cycle c_t is a stationary ARMA (p, q) process. Following previous literature (see Clark (1987), Morley et al. (2003), for instance), we set $p = 2$ and $q = 0$ so that the cyclical component of GDP takes the form

$$(1 - \phi_1 L - \phi_2 L^2)c_t = \varepsilon_t \tag{8}$$

Our estimation model consists of equations (4), (5), (7), and (8). Note that we follow MNZ and do not restrict the covariance $\sigma_{\eta\varepsilon}$ to be zero. With our FFF specification using $p = 2$ and $q = 0$, the model is identified in a similar fashion to that in MNZ because introducing the Fourier components introduces as many parameters in the UC_UR representation as in the reduced-form model. The model is cast in the State Space form and estimated by applying the Kalman Filter algorithm.

Notice that the linear specification for the trend is nested in the FFF model; in the absence of structural change, all values of α_k and β_k in equation (5) are equal zero. However, as a practical matter, it is not desirable to use a large value of n in the model. The use of many frequency components uses degrees of freedom and can lead to an over-fitting problem. Moreover, as shown in Becker, Enders and Hurn (2006), breaks are expected to manifest themselves at the low end of the spectrum. Thus, instead of positing the specific form of $\alpha(t)$, the issue is to select the proper frequencies to include in (2).

[1] Obviously, since each frequency component entails the estimation of two coefficients, if $n = T/2$, there are no available degrees of freedom and the fit of the model is perfect.

One way to select the frequency components is to do a grid search across the various values of k and select those frequency components resulting in the smallest value of some model selection criteria such as the AIC or BIC. The difficulty with this method is that testing for linearity becomes difficult since the various values of k are not identified under the null hypothesis that the values of α_i and $\beta_i = 0$. An alternative is to use the method suggested by Gallant and Souza (1991), Bierens (1997) and Becker, Enders, and Lee (2006), and use cumulative frequencies. The idea is to preselect some fixed value of n and use all frequency components in the interval 1 to n. Given n, the test for linearity (i.e., $\alpha_1 = \alpha_2 = \ldots = \alpha_n = \beta_1 = \beta_2 = \ldots \beta_n$) is asymptotically multivariate normal. As discussed in more detail below, we use a combination of both methods. Specifically, we first select $n = 1$ and test the restriction $\alpha_1 = \beta_1 = 0$. Since we are able to reject linearity in all cases, we go on to model the form of the trend by selecting other candidate frequencies using the BIC.

4. Empirical Results

4.1 Using MNZ data set

We begin by estimating the FFF model using the exact same data set as MNZ and Perron and Wada (2005), namely the quarterly data of log U.S. real GDP for the period 1947:Q1-1998:Q2. Following MNZ, the variable y_t in equation (4) is log real GDP times 100 so that the measure of the cycle may be read as the percentage deviation from trend.

Even though there are a number of reasons to allow for possible smooth structural changes in the trend function of real GDP, the key econometric issue is to determine whether it is possible to reject the null hypothesis of linearity. The most direct method is to test the null hypothesis of $n = 0$ against the alternative of $n = 1$ using a likelihood ratio (LR) test.[2] The LR statistic obtained for the period 1947:Q1-1998:Q2 is 8.58. With the degrees of freedom equal to two, the 5% critical value is 5.99 and the 1% critical value is 9.21. The absence of structural change in the trend is rejected at the 5% significance level.

Given the detection of structural change in the trend, the next step is to determine the most appropriate frequencies to include in equation (5). We rely on the standard model selection criteria BIC to select the frequency components to be included in the model. We estimate the model using different frequency components and obtain the *BIC* from each. It is well-known that structural breaks should manifest themselves at the low end of the spectrum. We found that frequencies higher than four did not perform as well as low frequency components. The *BIC* suggests that using frequencies 1 and 3 gives the best fit for the period 1947:Q1-1998:Q2. The last row of Table 1 reports the *BIC* of our model using $k = 1$ and 3 and the *BIC* of the UC-UR model (i.e., the unobserved components model with $k = 0$). Using $k = 1$ and 3 yields the smallest *BIC* = 2.795. The *BIC* of the UC-UR model is 2.814. (To save space, the values of *BIC* obtained from other frequency components are not reported here. They are available upon request.)

The second and the third columns of Table 1 present the maximum likelihood estimates of the UC-UR model and those of our model using $k = 1$ and 3, respectively. The estimates of the UC-UR model are very close to those reported in MNZ. $\hat{\sigma}_\eta^2$, which is the estimated variance of the trend innovation, is large and highly significant. It suggests that the trend is an $I(1)$ process. The estimated variance of the cycle innovation $\hat{\sigma}_\varepsilon^2$ is noticeably smaller than $\hat{\sigma}_\eta^2$. The trend and

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[2] If the various frequencies are estimated, each k is a nuisance parameter that is unidentified under the null hypothesis $\alpha_k = \beta_k = 0$. As suggested in Davis (1977) and Davis (1987), test statistics involving unidentified nuisance parameters may have non-standard distributions. However, the nuisance parameter problem is not a concern when testing the null of $n = 0$ against the alternative of $n = 1$ since k is given.

cycle innovations are strongly negatively correlated. The AR parameters ϕ_1 and ϕ_2 imply fairly low persistence in the cycle process. As is shown in the third column, the introduction of the Fourier components in the model generates remarkably different estimation results. First, the estimated variance of the trend innovation, $\hat{\sigma}_\eta^2$, is zero, which implies that the trend is deterministic once the unknown breaks are accounted for. In other words, accounting for structural breaks eliminates most of the persistence in the trend process. Second, the covariance of the trend and cycle innovations, $\hat{\sigma}_{\eta\varepsilon}$ = 0.438, is positive. Third, the cycle process is more persistent than that of the UC-UR model. The estimated AR coefficients are ϕ_1 = 1.509 and ϕ_2 = -0.615. The sum of the AR coefficients is $\phi_1 + \phi_2$ = 0.894, which suggests a fairly high persistence in the cycle.

We compare the trend-cycle decomposition obtained form our model with those obtained from three other models: MNZ's UC-UR model, the HP filter, and the PW model. Figure 3 shows our model compared to the UC-UR model. The shape of our trend suggests multiple smooth breaks. In addition to the productivity slowdown in the 1970s, our trend also captures a productivity resumption in the late 1990s and a productivity slowdown in the late 1950s. The trend is fairly smooth except for the breaks. By contrast, the UC-UR trend is very volatile and does not have obvious patterns. Our cycle corresponds very closely to the NBER chronology, while the UC-UR cycle is small and noisy and bears little resemblance to the NBER chronology. From Figure 3, we can see that introducing the Fourier components into the trend improves the decomposition remarkably.

Figure 4 shows our model compared to the HP filter using the standard value of λ = 1600 for quarterly data. The HP trend is smoother than our trend during the 1970s. Actually, the HP trend completely misses the productivity slowdown in the 70s. Instead, it shows a productivity slowdown in the early 80s. As a result, the HP cycle is different in magnitude and pattern from our cycle. For example, our cycle suggests a more severe recession in 1982, and our cycle identifies the mid-80s as a period of below-trend activity while the HP cycle identifies the mid-80s as a period of above-trend activity. This illustrates the point that a misspecification of the trend will also manifest itself as a misspecification of the cycle.

Figure 5 shows the PW decomposition as compared to our decomposition. Since it uses only a sharp break at 1973:Q1, the PW trend cannot capture other breaks such as the productivity resumption in the 1990s.

Figure 6 also illustrates the point that different trend specifications affect the patterns of the implied cycle. The panels on the left-hand side of Figure 6 show the density histograms of the cycle series obtained from different models over the sample period 1947:Q1-1998:Q2. The UC-UR cycle has a skewness of -0.02 with a significance level of 0.89. The cycle from our model has a skewness of -0.11 with a significance level of 0.55. Both of these two models produce symmetric cycles. The PW cycle is obviously more asymmetric. Its skewness is -0.29 with a significance level of 0.10. These results suggest that models with highly volatile trend, such as the UC-UR model, tend to attribute most fluctuations in the GDP series to the trend and produce symmetric cycle series. On the contrary, models with very smooth trend, such as the PW model, tend to produce an asymmetric cycle.

4.2 Using the Most Recent Data

We think that it would be of interest to update our study to the most recent data, especially since the MNZ data set does not cover the most recent recession, the 2001 recession. As such, we re-estimate our model using the quarterly data of log U.S. real GDP over 1947:Q1-2006:Q2. We also obtain the decompositions from the UC-UR model, the PW model, and the HP filter.

As in section 4.1, we test the null of $n = 0$ against the alternative of $n = 1$ using the likelihood ratio (LR) test. The LR statistic obtained for the period 1947:Q1-2006:Q2 is 9.16. Again, the absence of structural change in the trend is rejected at the 5% significance level. Next, we estimate the model using different frequency components and use the *BIC* to choose the most appropriate frequencies to include in the trend function. Using $k = 1$ yields the smallest BIC = 2.674, while the *BIC* of the UC-UR model is 2.677.

The fourth and the fifth columns of Table 1 report the estimation results of the UC-UR model and those of our model using $k = 1$. The results do not differ in any significant ways from those using MNZ data set. The UC-UR model suggests that the trend is an $I(1)$ process, the trend and cycle innovations are negatively correlated, and that the cycle exhibits low persistence. In contrast, our model suggests that the trend is deterministic once unknown breaks are accounted for, the trend and cycle innovations are positively correlated, and that the cycle exhibits very high persistence.

Figure 7 shows our decomposition compared to the decomposition from the UC-UR model. As before, the UC-UR trend is very volatile and the UC-UR cycle is small and noisy and bears little resemblance to the NBER chronology. Our trend here appears to be smoother than the one obtained using the MNZ data set. However, the shape of our trend still suggests multiple breaks: a productivity slowdown in the 70's and a productivity resumption in the late 1990's or early 2000's. Our cycle corresponds very closely to the NBER chronology.

Figure 8 shows our model compared to the HP filter and Figure 9 shows our model compared to the PW model. These two figures show some interesting result: the cycle from our model suggests a slower recovery from the 2001 recession than the HP cycle and the PW cycle do. As is shown in Figure 8, the HP cycle suggests that the economy was recovering quickly since 2003 and that the real GDP rose above the trend by 2005. However, the cycle from our model shows that the recover was slow and that the real GDP stayed below trend after 2001. The PW cycle in Figure 9 also shows a much faster recovery from the 2001 recession compared to our cycle. These differences in implied cycles are due to the failure of the HP filter and the PW model to capture the productivity resumption in the trend in the early 2000's. As such, they impute much increase in real GDP to cyclical increases rather than to a resumption in trend growth.

The right side of Figure 6 shows the density histograms of the cycle series obtained from different models over 1947:Q1-2006:Q2. As before, the implied cycle from the UC-UR model exhibits symmetry: its skewness is 0.15 with a significance level of 0.34. By contrast, the cycle series from our model and the PW model exhibits asymmetry: the implied cycle from our model has a skewness of -0.41 with a significance level of 0.01, and the implied cycle from the PW model has a skewness of -0.36 with a significance level of 0.02. Those results reinforce the point that models with highly volatile trend to attribute most fluctuations in the GDP series to the trend and produce symmetric cycle series, while models with smoother trend tend to do the opposite.

5. Conclusion

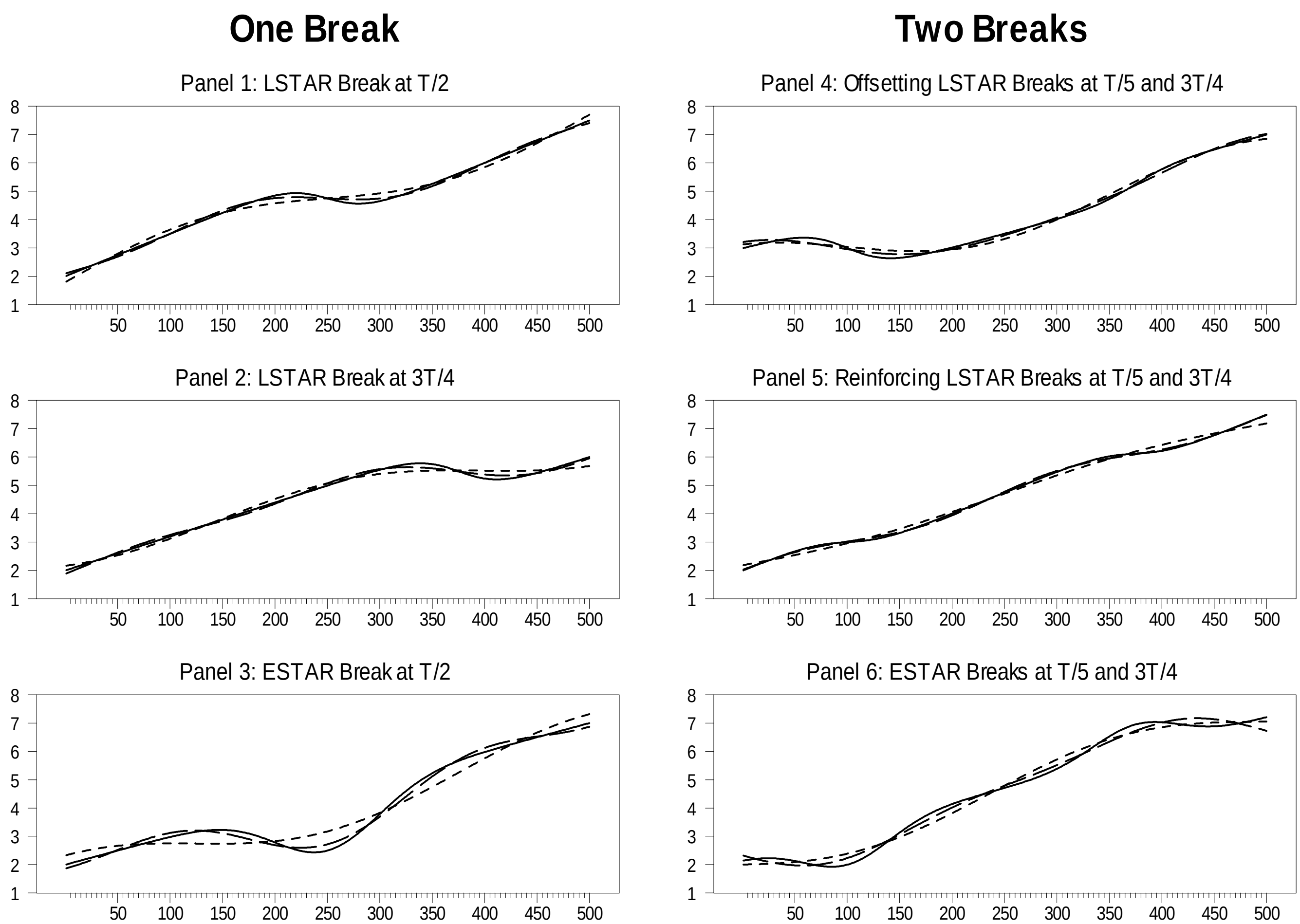
Essential characteristics of one or more structural breaks can be captured using a small number of low frequency components from a Fourier approximation. In this paper, we introduce the Fourier series into the trend function of U.S. real GDP to allow for gradual effects of unknown numbers of structural breaks occurring at unknown dates. We find that the trend is deterministic once we account for unknown breaks. Our trend captures multiple breaks. In addition to the productivity slowdown in the 1970s, our trend also captures a productivity resumption in the late 1990s and early 2000s and a slowdown in the late 1950s. The trend is fairly smooth except for these breaks. Our cycle corresponds very closely to the NBER chronology. We compare the decomposition from our model with those from the UC-UR model, the HP filter, and the PW model. We find that our decomposition has several favorable characteristics over theirs. For example, the cycle from our model suggests a slower recovery from the 2001 recession than the HP filter and the PW model, which is due to the incapability of the HP filter and the PW model to capture the productivity resumption in the trend in the early 2000s. We also find that the specification of the trend greatly influences the symmetry of the cycle series. Models with highly volatile trend tend to attribute most fluctuations in the GDP series to the trend and produce symmetric cycle series, while models with smoother trend tend to attribute most fluctuations in the GDP series to the cycle and produce asymmetric cycle series.

Table 1. Maximum Likelihood Estimates and the *BIC*

	1947:Q1- 1998:Q2		*1947:Q1- 2006:Q2*	
	UC-UR	***k* =1, 3**	**UC-UR**	***k* = 1**
μ	0.827 (0.086)	0.819 (0.012)	0.852 (0.077)	0.840 (0.012)
σ_η	1.233 (0.162)	0.000 (0.079)	1.130 (0.116)	0.000 (0.108)
σ_ε	0.707 (0.319)	0.331 (0.285)	0.620 (0.185)	0.349 (0.212)
$\sigma_{\eta\varepsilon}$	-0.792 (0.421)	0.438 (0.147)	-0.633 (0.247)	0.378 (0.113)
ϕ_1	1.351 (0.150)	1.509 (0.124)	1.345 (0.127)	1.527 (0.111)
ϕ_2	-0.724 (0.169)	-0.615 (0.113)	-0.736 (0.109)	-0.596 (0.106)
α_1	N/A	0.181 (0.018)	N/A	0.111 (0.021)
β_1	N/A	0.004 (0.031)	N/A	0.037 (0.033)
α_3	N/A	0.162 (0.056)	N/A	N/A
β_3	N/A	0.164 (0.067)	N/A	N/A
BIC	2.814	2.795	2.678	2.674

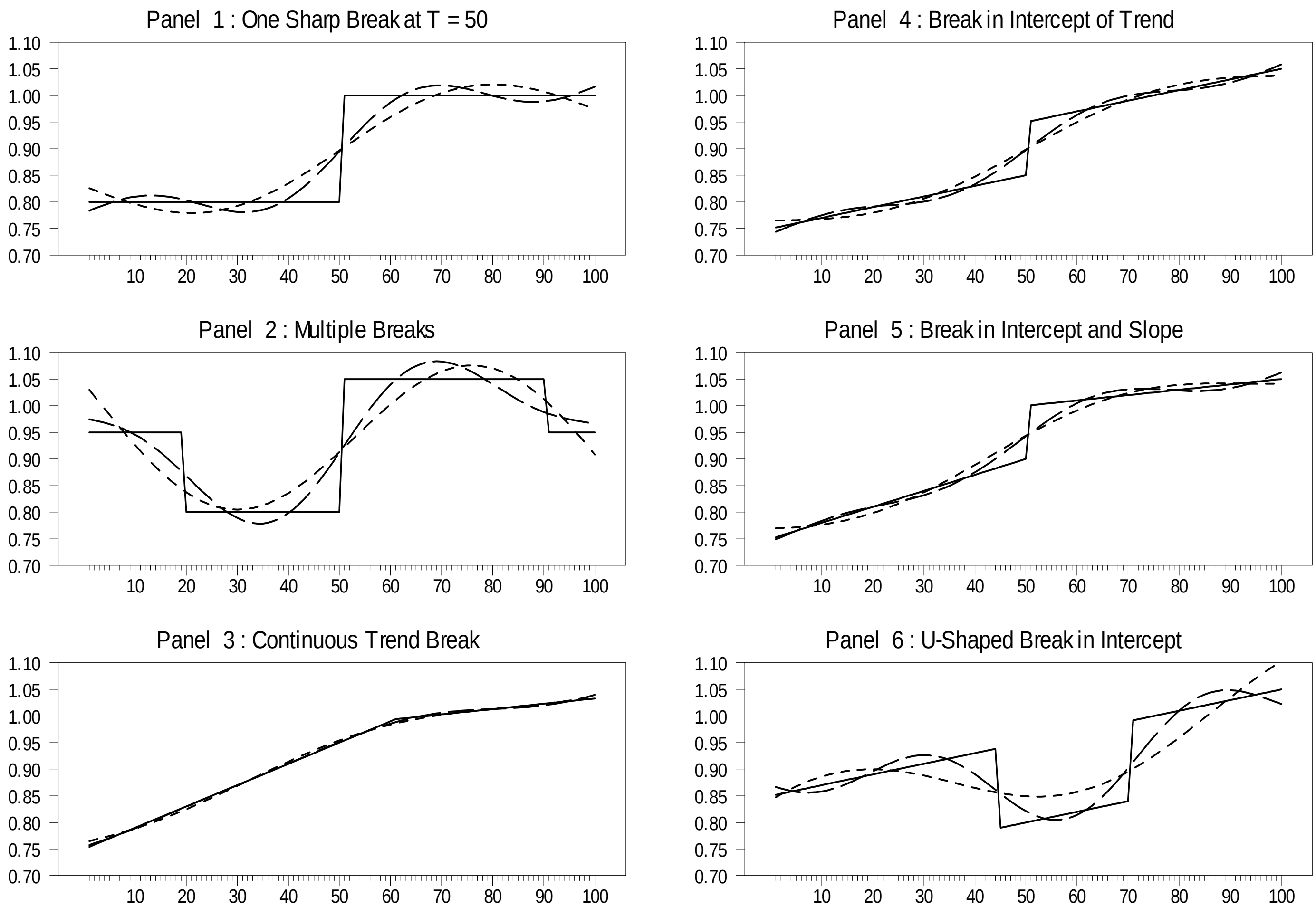
Note: standard errors are in parentheses.

Figure 1. Fourier Approximation of Smooth Breaks



Series: ____ 1-Frequency: _ _ _ 2-Frequencies: __ __

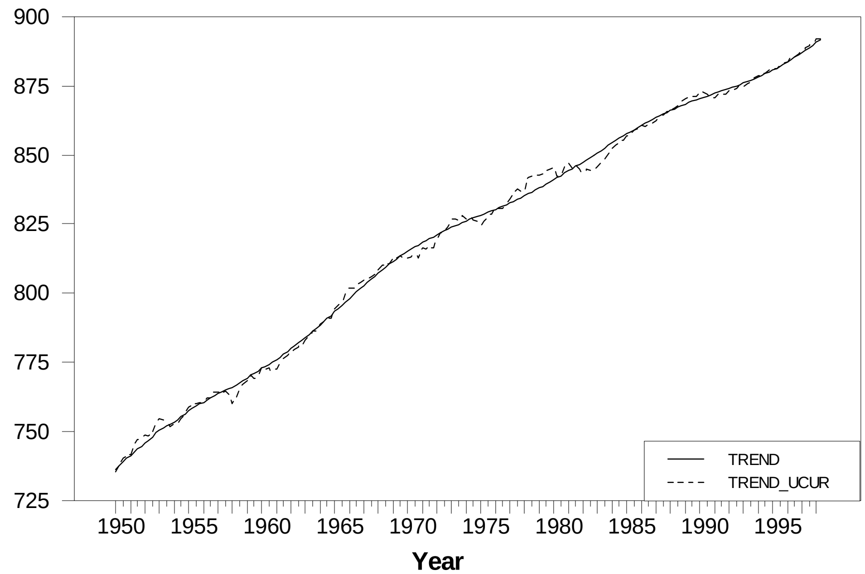
Figure 2. Fourier Approximation of Sharp Breaks



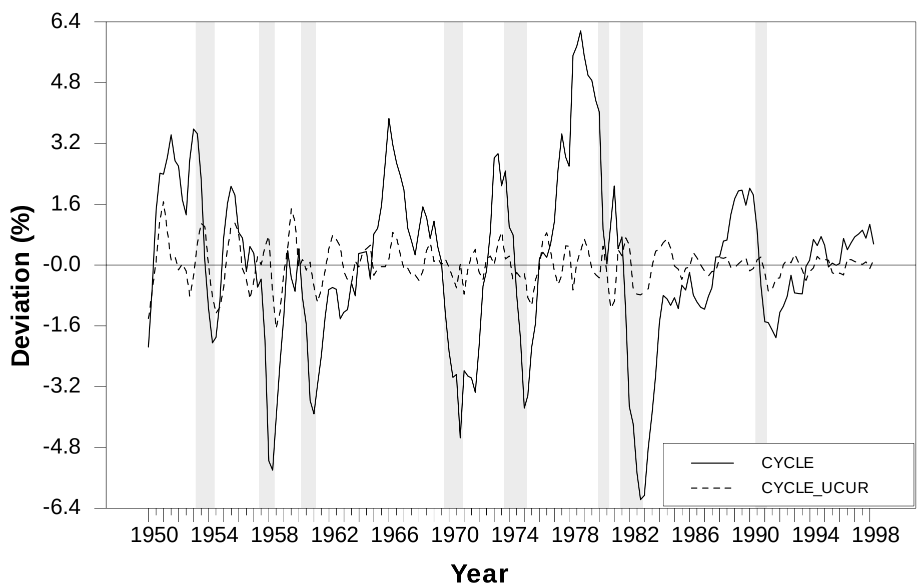
Series: ____ 1-Frequency: _ _ _ 2-Frequencies: __ __

Figure 3. Our Model Compared to the UC-UR Model for 1947:Q1-1998:Q2

Trend



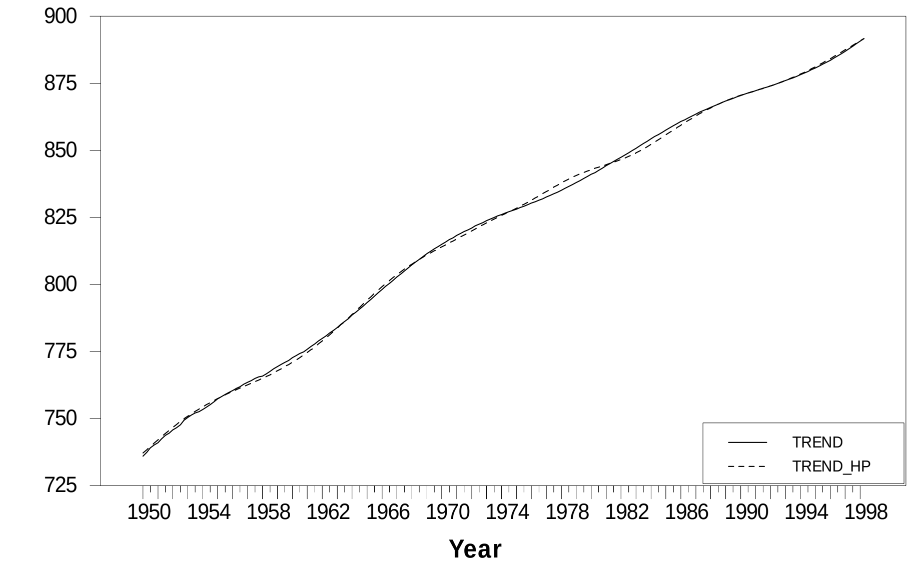
Cycle



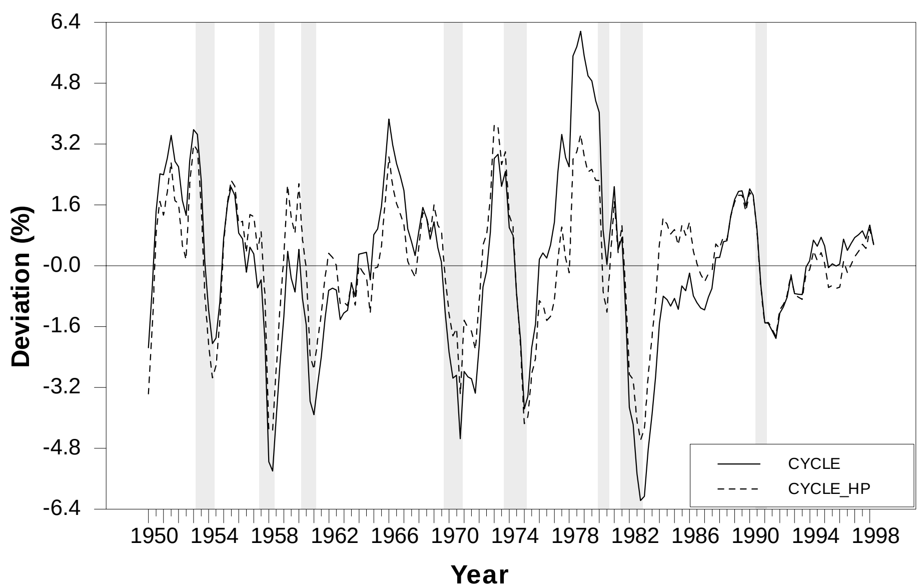
Note: TREND and CYCLE represent the decomposition obtained from our model. TREND_UCUR and CYCLE_UCUR represent the decomposition obtained from the UC-UR model.

Figure 4. Our Model Compared to the HP Filter for 1947:Q1-1998:Q2

Trend



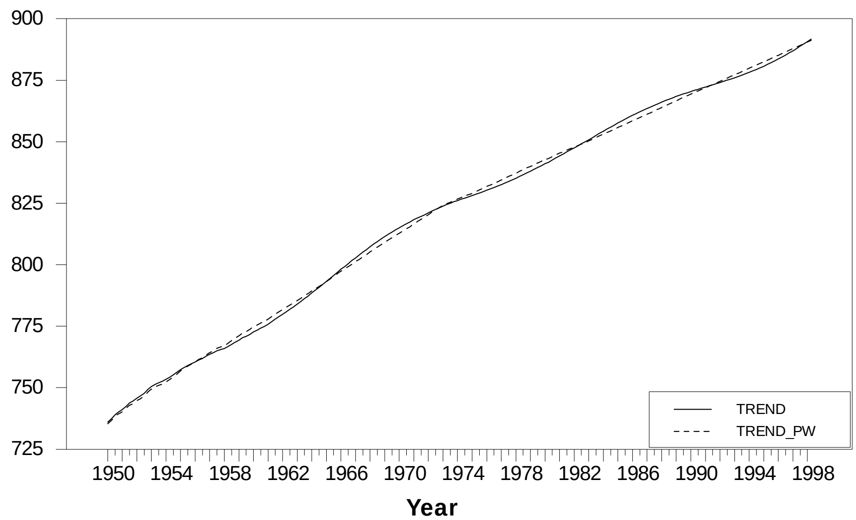
Cycle



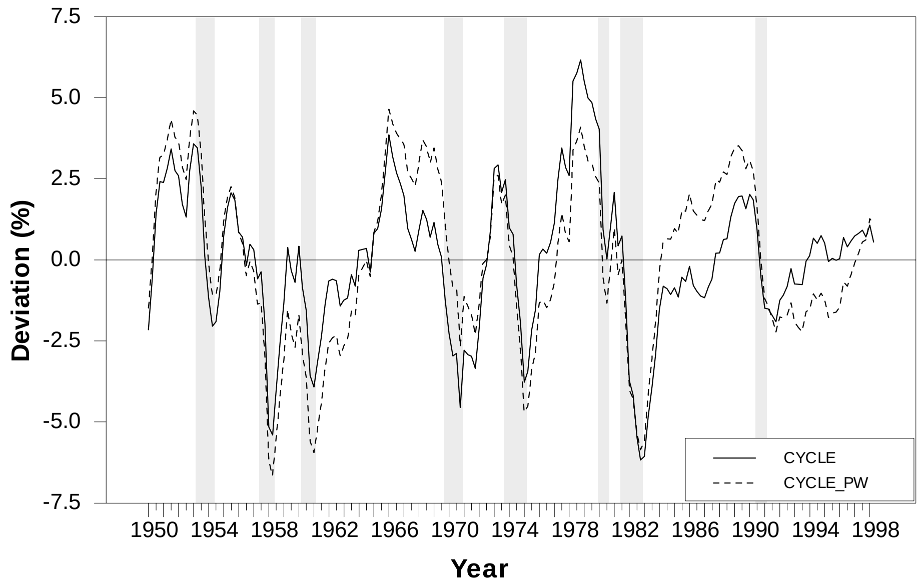
Note: TREND and CYCLE represent the decomposition obtained from our model. TREND_HP and CYCLE_HP represent the decomposition obtained from the HP filter.

Figure 5. Our Model Compared to the PW Model for 1947:Q1-1998:Q2

Trend

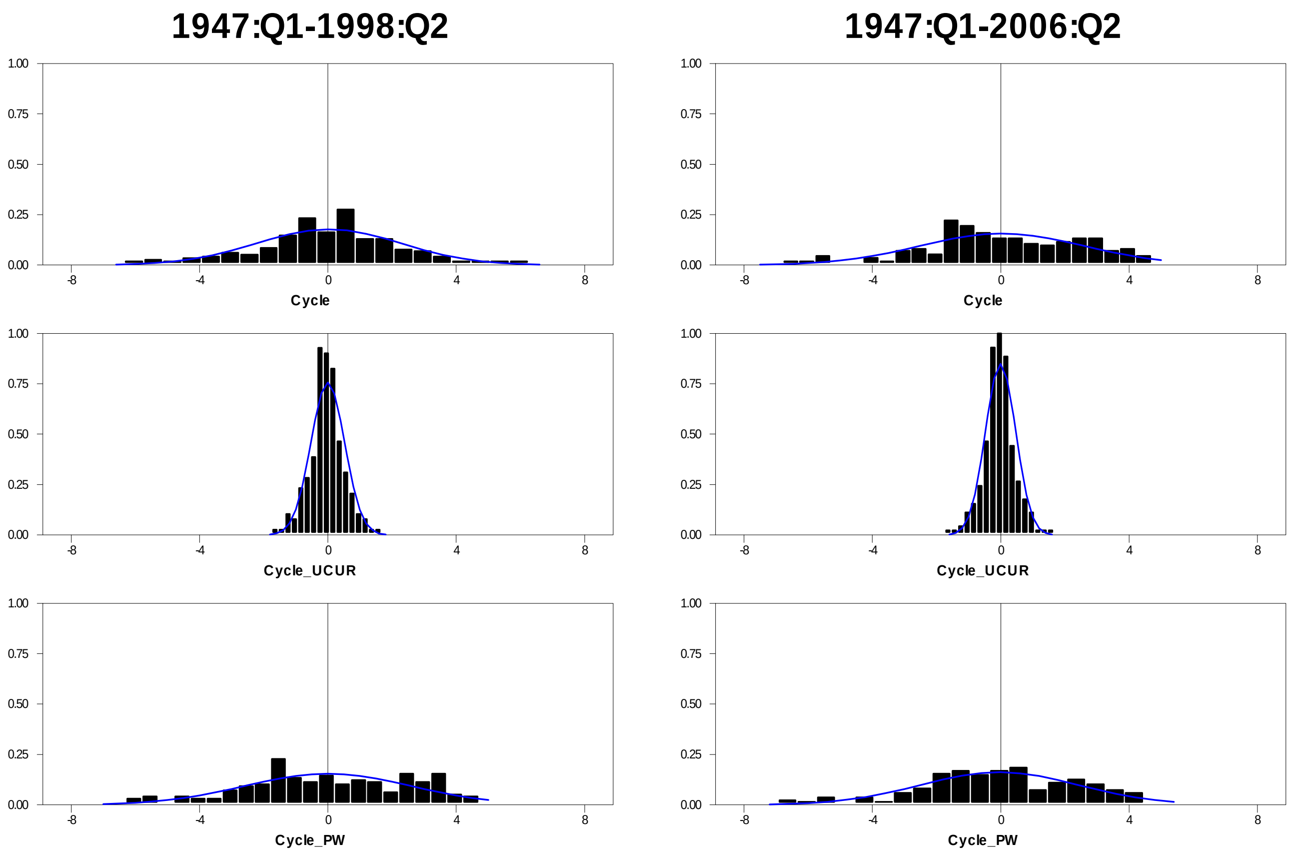


Cycle



Note: TREND and CYCLE represent the decomposition obtained from our model. TREND_PW and CYCLE_PW represent the decomposition obtained from the PW model.

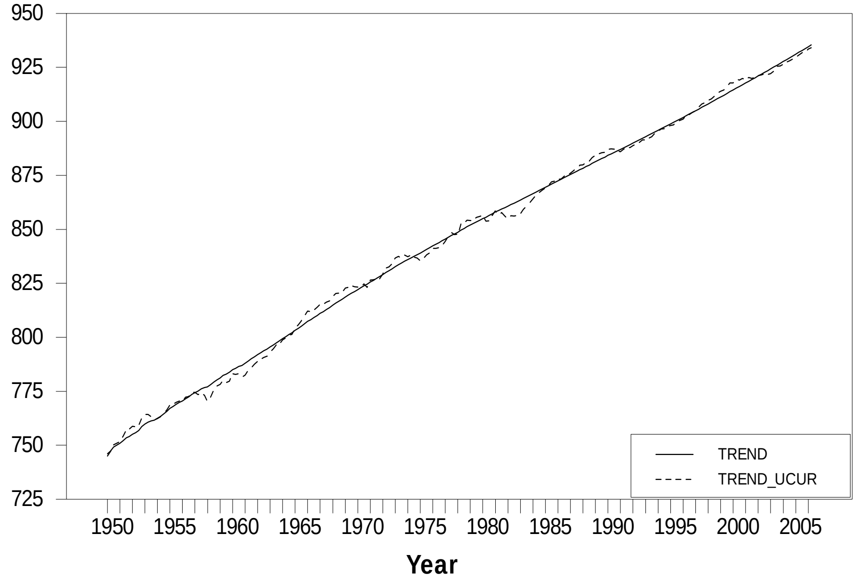
Figure 6. The Density Histograms of the Cycle Series for the two sample periods



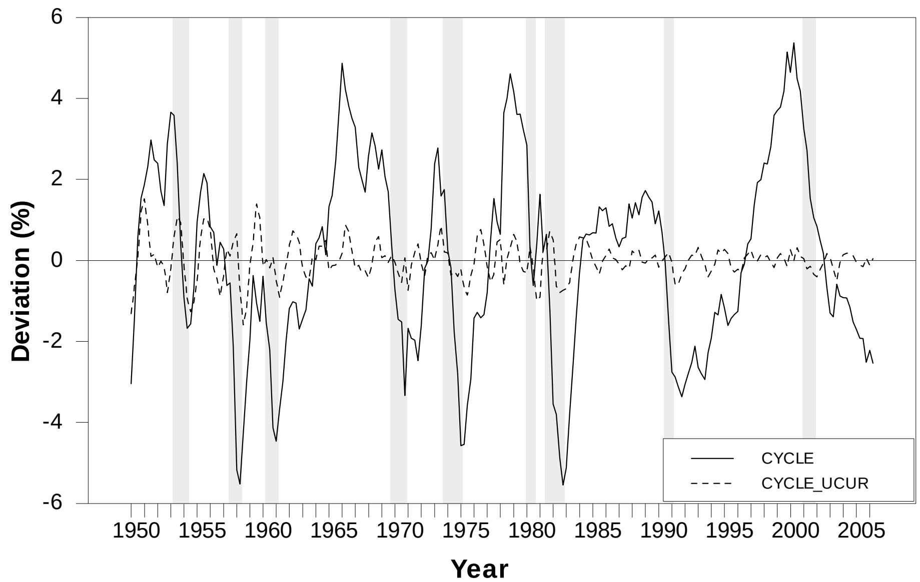
Note: Cycle, Cycle_UCUR, and Cycle_PW represent the implied cycle from our model, from the UC-UR model, and from the PW model, respectively. Each graph displays an overlay of the density curve of the normal distribution.

Figure 7. Our Model Compared to the UC-UR Model for 1947:Q1-2006:Q2

Trend



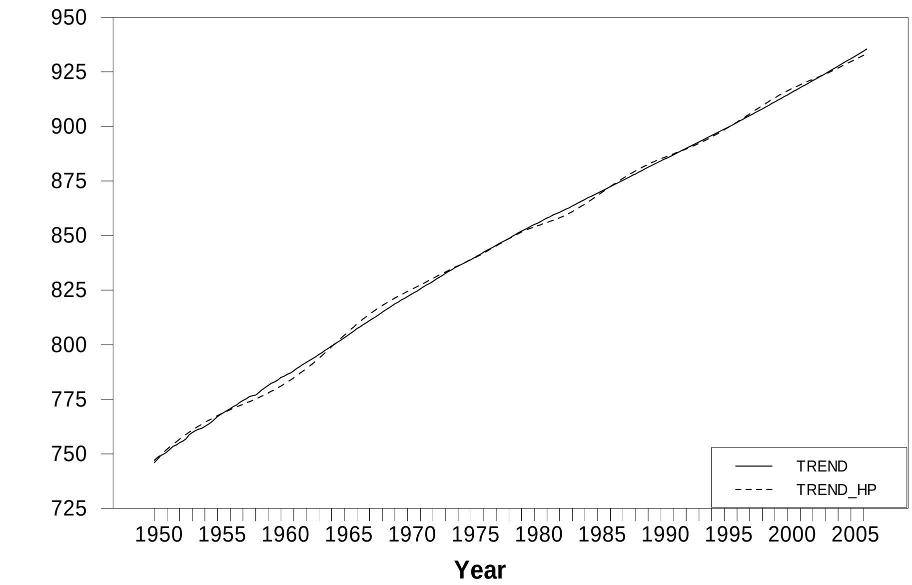
Cycle



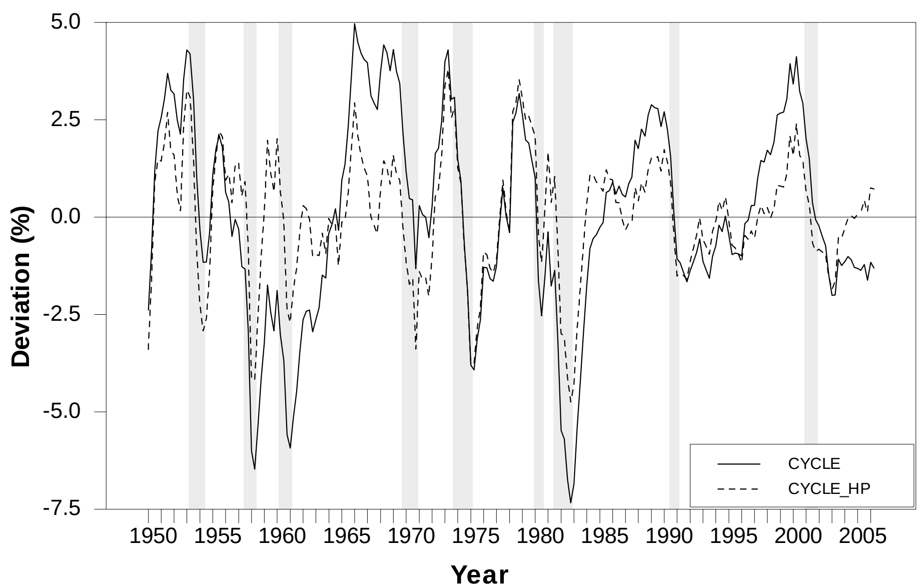
Note: TREND and CYCLE represent the decomposition obtained from our model. TREND_UCUR and CYCLE_UCUR represent the decomposition obtained from the UC-UR model.

Figure 8. Our Model Compared to the HP Filter for 1947:Q1-2006:Q2

Trend



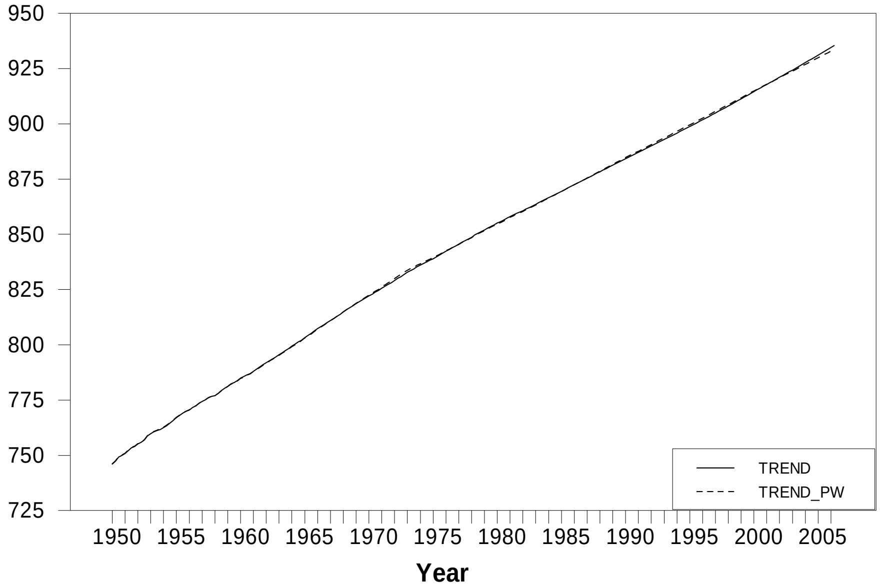
Cycle



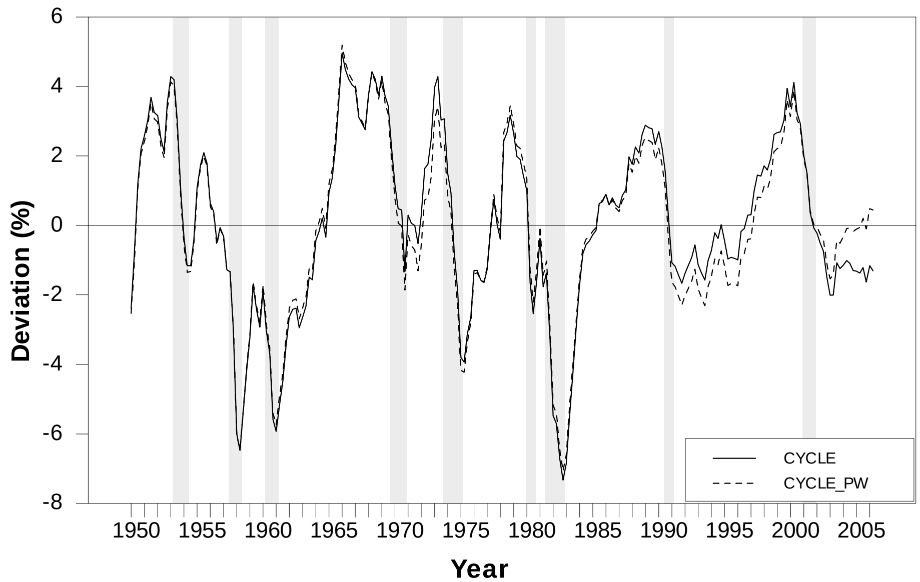
Note: TREND and CYCLE represent the decomposition obtained from our model. TREND_HP and CYCLE_HP represent the decomposition obtained from the HP filter.

Figure 9. Our Model Compared to the PW Model for 1947:Q1-2006:Q2

Trend



Cycle



Note: TREND and CYCLE represent the decomposition obtained from our model. TREND_PW and CYCLE_PW represent the decomposition obtained from the PW model.

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